THE VANISHING OF THE THETA FUNCTION IN THE KP DIRECTION: A GEOMETRIC APPROACH

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ABSTRACT. We give a geometric proof of a formula, due to Segal and Wilson, which describes the order of vanishing of the Riemann theta function in the direction which corresponds to the direction of the tangent space of a Riemann surface at a marked point. While this formula appears in the work of Segal and Wilson as a by-product of some non-trivial constructions from the theory of integrable systems (loop groups, infinite-dimensional Grassmannians, tau functions, Schur polynomials, ...) our proof only uses the classical theory of linear systems on Riemann surfaces.

CONTENTS

1. INTRODUCTION

The fundamental paper [SW] by Segal and Wilson on soliton equations leads to an explicit formula for computing the vanishing of the Riemann theta function in a direction which is natural from the geometric point of view. In order to present this formula, let C be a compact Riemann surface (of genus $g > 1$), let $p \in C$ and let Θ denote the theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$. Also let ϑ denote Riemann's theta function, for which $(\vartheta) = \Theta$. For a point $\mathcal{L} \in \Theta$, consider the embedding $C \to \mathrm{Pic}^{g-1}(C) : q \mapsto \mathcal{L}(q - p)$. The natural direction alluded to above is the tangent space X_p to this embedded curve at $\mathcal{L}$. Following [SW] the vanishing of ϑ at $\mathcal{L}$ in the direction of X_p, denoted $\mathrm{ord}_{\mathcal{L}}(\vartheta, X_p)$ is obtained by considering the infinite subset of $\mathbb{Z}$, defined by

$$S_{\mathcal{L}} = \{s \in \mathbb{Z} \mid h^0(\mathcal{L}((s + 1)p)) = h^0(\mathcal{L}(sp)) + 1\}.$$

In fact, the sections of $\mathcal{L}$ over $C \setminus \{p\}$ define an infinite-dimensional plane W, which is an element of the Sato Grassmannian and the vanishing of the tau function in the KP-direction is, according to [SW, Prop. 8.6], given by the codimension of W,

which is explicitly given by the finite sum $\sum_{i \geq 0} i - s_i$, upon writing $S_{\mathcal{L}} = \{s_0 < s_1 < s_2 < \cdots\}$. The tau function coincides, up to an exponential factor, with the Riemann theta function of C ([SW, Th. 9.11]) and the tangent direction X_p coincides with the KP-direction (see [S, Lemma 5 and Appendix 0]). Therefore, the order of vanishing is given by

$$\operatorname{ord}_{\mathcal{L}}(\vartheta, X_p) = \sum_{i \geq 0} i - s_i.$$

The purpose of this paper is to give an algebraic-geometric proof of this result. Our proof uses (only) the classical theory of linear systems on Riemann surfaces and it highlights the geometric meaning of the order of vanishing. As is pointed out in [SW, footnote p. 51] an independent (analytical) proof of this formula has also been given by John Fay (see [F]), by using the theory of theta functions.

The first step of our approach consists of an interpretation of the order of vanishing as the intersection multiplicity of the theta divisor with a copy of C, properly embedded (at least around $\mathcal{L}$) in $\operatorname{Pic}^{g-1}(C)$. If we pull back the theta divisor using this embedding we find a divisor R on C which is the sum of the ramification divisors of the maps

$$\varphi_{\mathcal{L}}^k : C \longrightarrow \operatorname{Grass}_{k+1}(H^0(\mathcal{L})^*),$$

which are the natural generalizations of the morphisms $\varphi_{\mathcal{L}} : C \longrightarrow \mathbb{P}(H^0(\mathcal{L})^*)$ defined by the linear system $|\mathcal{L}|$ (assumed here base point free). It follows that the order of vanishing is given by the multiplicity of p in R, leading to

$$\operatorname{ord}_{\mathcal{L}}(\vartheta, X_p) = \sum_{i=1}^{n} m_i - i.$$

where $\{m_1 < \cdots < m_n\}$ is the gap sequence $G_p(\mathcal{L}(np))$ of $\mathcal{L}(np)$ at p. This formula is independent of n, which is assumed sufficiently large (e.g. $n = g$ will do). Noticing that for $n = g$ one has $s_i = g - m_{g-i}$ (for $i = 0, \ldots, g-1$) from which Formula (1) follows at once.

Notice that the Segal-Wilson formula for the vanishing of the tau function may also be applied in the case of tau functions that come from singular curves. It would be interesting to adapt our geometric arguments to this case, leading to a formula for the vanishing of the *theta functions for singular curves*, as proposed in [SW, Remark 6.13].

The structure of this paper is as follows. In Section 2 we fix the notation and we recall the notions of gap numbers for arbitrary line bundles. In Section 3 we translate the order of vanishing of the theta function in terms of intersection theory and we show that this order is given as an inflectionary weight. This is used in Section 5 to obtain an explicit formula, which we show to be equivalent to the formula by Segal and Wilson.

2. PRELIMINARIES

In this section we introduce the notation and collect some results on curve theory. Throughout the whole paper C denotes a compact Riemann surface of genus g and $p \in C$ a marked point.

For a divisor D on C we denote by $\mathcal{O}_C(D)$ the corresponding line bundle and for a line bundle $\mathcal{L}$ on C its linear system is denoted by $|\mathcal{L}|$. We use the standard abbreviations $h^0(\mathcal{L})$ for $\dim H^0(C, \mathcal{L})$ and $\mathcal{L}(D)$ for $\mathcal{L} \otimes \mathcal{O}_C(D)$, where $\mathcal{L}$ is any line

bundle and D is any divisor on C. We will use the Riemann-Roch theorem in the form

$$h^0(\mathcal{L}) = h^0(\omega_C \otimes \mathcal{L}^{-1}) - g + \deg(\mathcal{L}) + 1,$$

where $\mathcal{L}$ is any line bundle on C and ω_C is the canonical bundle of C.

We now recall the notions of gap numbers and inflectionary weights. For proofs and details we refer to [Mi, Sect. VII.4] and to [ACGH, Ch. 1 Ex. C].

Let $\mathcal{L}$ be a line bundle on C of positive degree and let $q \in C$. An integer $m \geq 1$ is called a *gap number for $\mathcal{L}$ at q* if

$$h^0(\mathcal{L}(-mq)) = h^0(\mathcal{L}(-(m-1)q)) - 1,$$

and the set $G_q(\mathcal{L})$ of gap numbers for $\mathcal{L}$ at q is called the *gap sequence of $\mathcal{L}$ at q*; its cardinality is $r = h^0(\mathcal{L})$ and no gap number is larger than $\deg \mathcal{L} + 1$. Writing

$$G_q(\mathcal{L}) = \{1 \leq m_1 < m_2 < \cdots < m_r \leq \deg \mathcal{L} + 1\},$$

we have that $m_1 > 1$ if and only if q is a base point of $\mathcal{L}$ and that $m_r = \deg \mathcal{L} + 1$ if and only if $\mathcal{L} = \mathcal{O}_C(\deg \mathcal{L} \cdot q)$. For a general point $q \in C$ the gap sequence of $\mathcal{L}$ at q is $\{1, 2, \ldots, h^0(\mathcal{L})\}$; a point q for which the gap sequence of $\mathcal{L}$ at q is not of this form is called an *inflection point for $\mathcal{L}$*. Notice that q is an inflection point if and only if $h^0(\mathcal{L}(-rq)) \neq 0$, where $r = h^0(\mathcal{L})$.

If the linear system $|\mathcal{L}|$ is base point free the inflection points have the following geometric interpretation. Consider the morphism $\varphi_{\mathcal{L}} : C \longrightarrow \mathbb{P}(H^0(\mathcal{L})^*)$ defined by the linear system $|\mathcal{L}|$. For a generic $q \in C$ there is a unique k-dimensional osculating plane to $\varphi_{\mathcal{L}}(C)$ at $\varphi_{\mathcal{L}}(q)$, yielding a well-defined morphism

$$(1) \qquad \varphi_{\mathcal{L}}^k : C \longrightarrow \mathrm{Grass}_{k+1}(H^0(\mathcal{L})^*),$$

called the *k-th associated map*. This way one arrives at $h^0(\mathcal{L}) - 1$ associated maps $\varphi_{\mathcal{L}}^{i-1}$, $i = 1, \ldots, h^0(\mathcal{L}) - 1$, ($\varphi_{\mathcal{L}}^0 = \varphi_{\mathcal{L}}$). In these terms a point q is an inflection point if and only if q is a ramification point of one of the maps $\varphi_{\mathcal{L}}^k$. We denote the ramification divisor of $\varphi_{\mathcal{L}}^k$ by $R_k(\mathcal{L})$ and we define

$$R(\mathcal{L}) = \sum_{k=1}^{h^0(\mathcal{L})-1} R_{k-1}(\mathcal{L}).$$

The multiplicity $w_q(\mathcal{L})$ of q in $R(\mathcal{L})$ is called the *inflectionary weight* of q with respect to $\mathcal{L}$ and is given by

$$(2) \qquad w_q(\mathcal{L}) = \sum_{i=1}^{h^0(\mathcal{L})} (m_i - i).$$

When $\mathcal{L}$ is not base point free we define the inflectionary weights $w_q(\mathcal{L})$ by (2) and the ramification divisor $R(\mathcal{L})$ by $R(\mathcal{L}) = \sum_q w_q(\mathcal{L})q$. This divisor admits an alternative description as the zero divisor of $W = W(z)(dz)^{\frac{n(n-1)}{2}}$ where z is a local coordinate, $n = h^0(\mathcal{L})$ and $W(z) = W(f_1, \ldots, f_n)$ is the Wronskian with respect to any basis $f_1, \ldots, f_n$ of $H^0(\mathcal{L})$. In particular W is a holomorphic section of the line bundle $\mathcal{L}^n \otimes \omega_C^{\frac{n(n-1)}{2}}$.

Taking $\mathcal{L} = \omega_C$ one recovers the well-known notion of the gap sequence of $q \in C$, denoted by G_q, and the above definition of the inflectionary points and weights reduces, by a simple application of Riemann-Roch, to the standard definition of Weierstraß points and their weights.

Finally let us fix our conventions about the Jacobian $J(C)$ of C. By definition $J(C) = H^0(\omega_C)^*/H_1(C, \mathbb{Z})$, so that the vector space $H^0(\omega_C)^*$ is canonically identified with the tangent space of $J(C)$ at every point. By the Abel-Jacobi theorem there is a canonical isomorphism $J(C) \simeq \mathrm{Pic}^0(C)$. Moreover every line bundle $\mathcal{L}$ on C of degree $g-1$ induces an isomorphism $J(C) \simeq \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^{g-1}(C)$, $\mathcal{P} \mapsto \mathcal{L} \otimes \mathcal{P}$. For our purposes it is convenient to work with $\mathrm{Pic}^{g-1}(C)$ rather than $\mathrm{Pic}^0(C)$. So in the sequel we identify $J(C)$ with $\mathrm{Pic}^{g-1}(C)$ without further notice; the underlying isomorphism (respectively line bundle defining the isomorphism) will always be evident from the context. The main advantage working with $J(C) = \mathrm{Pic}^{g-1}(C)$ is, that we have a canonical *theta divisor* $\Theta = \{\mathcal{L} \in J(C) \mid h^0(\mathcal{L}) > 0\}$. By Riemann-Roch, Θ is invariant with respect to the natural involution

$$(3) \qquad\qquad \iota : J(C) \longrightarrow J(C), \ \iota(\mathcal{L}) = \omega_C \otimes \mathcal{L}^{-1}.$$

More precisely we have $h^0(\mathcal{L}) = h^0(\iota(\mathcal{L}))$ for any $\mathcal{L} \in \Theta$.

We denote by ϑ the Riemann theta function on $H^0(\omega_C)^*$ for which $\pi^*\Theta$ is the zero divisor of ϑ, where π is the natural projection $H^0(\omega_C)^* \to J(C)$.

For any $\mathcal{L} \in J(C)$ we have an embedding $\alpha_{\mathcal{L},p}$ of C into $J(C)$, given by $\alpha_{\mathcal{L},p}(q) = \mathcal{L}(q-p)$. Clearly for different $\mathcal{L}$ and p the maps $\alpha_{\mathcal{L},p}$ only differ by a translation on $J(C)$.

3. GEOMETRIC DESCRIPTION OF THE ORDER OF VANISHING

Let $\mathcal{L} \in J(C)$ and let X be a one-dimensional subvector space of the tangent space $H^0(\omega_C)^*$ at $\mathcal{L}$. Choose any point l in the fiber of π over $\mathcal{L}$ and consider the affine line $l + X$ which passes through l and which has direction X. The order of vanishing of $\vartheta|_{l+X}$ at the point l is independent of the choice of $l \in \pi^{-1}(\mathcal{L})$. So define the *order of vanishing of ϑ at $\mathcal{L}$ in the direction of X*, denoted $\mathrm{ord}_{\mathcal{L}}(\vartheta, X)$, as $\mathrm{ord}_l \vartheta|_{l+X}$. If Θ does not contain the straight line $\bar{X} = \pi(l+X)$ then there exists a small neighborhood U of l in $l + X$ such that $\pi(U) \cap \Theta = \{\mathcal{L}\}$ and $\mathrm{ord}_{\mathcal{L}}(\vartheta, X) = (\pi(U) \cdot \Theta)_{\mathcal{L}}$, the intersection multiplicity of Θ with $\pi(U)$ at $\mathcal{L}$.

Let X_p denote the tangent space to $\alpha_{\mathcal{L},p}(C)$ at $\mathcal{L}$. Notice that, as a subvector space of $H^0(\omega_C)^*$, X_p does not depend on $\mathcal{L}$ but only on the point $p \in C$. We wish to compute $\mathrm{ord}_{\mathcal{L}}(\vartheta, X_p)$ for an arbitrary $\mathcal{L} \in \Theta$; if $\mathcal{L} \notin \Theta$ then this order is trivially zero. It can be shown [1] that for all C, $\mathcal{L}$ and p that $\bar{X}_p$ is not contained in Θ, which is clearly true as soon as C, $\mathcal{L}$ or p is generic. For this the idea is to replace $\pi(U)$ by a a complete curve which, around $\mathcal{L}$, looks like $\pi(U)$. Notice that if $\mathcal{L} \in \Theta$ we cannot use $\alpha_{\mathcal{L},p}(C)$ because the latter does not necessarily intersect Θ properly. Consider for any integer $n \neq 0$ the morphism

$$(4) \qquad\qquad \alpha_{\mathcal{L},p,n} : C \longrightarrow J(C), \ \alpha_{\mathcal{L},p,n}(q) = \mathcal{L}(nq - np).$$

Notice that $\alpha_{\mathcal{L},p,n}(p) = \mathcal{L}$ and that for a small neighborhood V of p in C the tangent space to $\alpha_{\mathcal{L},p,n}(V)$ at $\mathcal{L}$ is precisely X_p.

Lemma 3.1. *For all $\mathcal{L} \in J(C)$ and $n > 0$ we have*

    (1) $\alpha_{\mathcal{L},p,n}(C)$ *and Θ intersect properly if and only if $h^0(\mathcal{L}(-np)) = 0$;*

    (2) $\alpha_{\mathcal{L},p,-n}(C)$ *and Θ intersect properly if and only if $h^0(\iota(\mathcal{L})(-np)) = 0$.*

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[1] This follows from [SW, Prop. 8.6 and Th. 9.11] but a geometric proof of this (geometric!) property is unknown.

Proof. We prove (1), the proof of (2) is similar. Recall that an irreducible curve intersects a divisor properly precisely when the curve is not contained in the support of the divisor. So $\alpha_{\mathcal{L},p,n}(C)$ and Θ do not intersect properly if and only if $h^0(\mathcal{L}(nq - np)) > 0$ for all $q \in C$. We claim that this is equivalent to $h^0(\mathcal{L}(-np)) > 0$. Indeed, by Riemann-Roch

$$h^0(\mathcal{L}(nq - np)) = h^0(\iota(\mathcal{L})(np - nq)), \quad \text{and}$$
$$h^0(\iota(\mathcal{L})(np)) = h^0(\mathcal{L}(-np)) + n.$$

So $h^0(\mathcal{L}(np - nq)) > 0$ for all $q \in C$ if and only if $h^0(\iota(\mathcal{L})(np)) > n$, leading to our claim. $\square$

In particular, when $|n| \geq g$ then

$$\operatorname{ord}_{\mathcal{L}}(\vartheta, X_p) = (\Theta \cdot \alpha_{\mathcal{L},p,n}(V))_{\mathcal{L}},$$

where V is a small neighborhood of p in C. Pulling this intersection back to C we get that for any $|n| \geq g$

$$\operatorname{ord}_{\mathcal{L}}(\vartheta, X_p) = \operatorname{mult}_p(\alpha_{\mathcal{L},p,n}^*\Theta).$$

This multiplicity will be computed in the next section.

4. The divisor $\alpha_{\mathcal{L},p,-n}^*\Theta$

The aim of this section is to prove the following

Theorem 4.1. *For all $\mathcal{L} \in J(C)$, $p \in C$, and $n > 0$ with $h^0(\iota(\mathcal{L})(-np)) = 0$*

$$\alpha_{\mathcal{L},p,-n}^*\Theta = R(\mathcal{L}(np)).$$

For the proof we need the following

Proposition 4.2. *For all $\mathcal{L} \in J(C), p \in C$ and $n > 0$*

$$\alpha_{\mathcal{L},p,-n}^*\mathcal{O}_{J(C)}(\Theta) = (\mathcal{L}(np))^n \otimes \omega_C^{\frac{n(n-1)}{2}}.$$

Proof.

STEP I: The case $n = 1$ follows exactly from [LB] Lemma 11.3.4 with $x = 0, \kappa = \mathcal{L}$, and $c = p$.

STEP II: For $n \geq 1$ consider the difference map

$$\delta_{\mathcal{L}}^n : C^{2n} \longrightarrow J(C), \quad \delta_{\mathcal{L}}^n(p_1, q_1, \ldots, p_n, q_n) = \mathcal{L}(\textstyle\sum_i p_i - q_i)$$

and denote by $\pi_i^n : C^{2n} \longrightarrow C$ the i-th projection. We show by induction on n that for all $n \geq 1$ and all $\mathcal{L} \in J(C)$

$$\delta_{\mathcal{L}}^{n*}\mathcal{O}_{J(C)}(\Theta) =$$

(5)
$$\bigotimes_{i=1}^{n}\left(\pi_{2i-1}^{n}{}^*(\omega_C \otimes \mathcal{L}^{-1}) \otimes \pi_{2i}^{n}{}^*\mathcal{L}\right) \otimes \mathcal{O}_{C^{2n}}\left(\sum_{1 \leq i < j \leq 2n}(-1)^{i+j+1}(\pi_i^n, \pi_j^n)^*\Delta\right),$$

where Δ denotes the diagonal in C^2.

For $n = 1$ we have to show that

$$\delta_{\mathcal{L}}^{1}{}^*\mathcal{O}_{J(C)}(\Theta) = \pi_1^{1}{}^*(\omega_C \otimes \mathcal{L}^{-1}) \otimes \pi_2^{1}{}^*\mathcal{L} \otimes \mathcal{O}_{C^2}(\Delta)$$

for all $\mathcal{L} \in \operatorname{Pic}^{g-1}(C)$. According to the Seesaw Principle (see [LB] A.9) it suffices to show that the restrictions to $C \times \{q\}$ and $\{q\} \times C$ of both sides of the equation

coincide for all $q \in C$. But since the composition of $\delta_{\mathcal{L}}^1$ with the natural embedding $C \simeq C \times \{q\} \longrightarrow C \times C$ is the map $\iota \circ \alpha_{\omega_C \otimes \mathcal{L}^{-1}, q, -1}$ and $\iota^* \Theta = \Theta$ we have, using Step I,

$$\delta_{\mathcal{L}}^{1*} \mathcal{O}_{J(C)}(\Theta) | C \times \{q\} = \alpha_{\omega_C \otimes \mathcal{L}^{-1}, q, -1}^* \mathcal{O}_{J(C)}(\Theta) = \omega_C \otimes \mathcal{L}^{-1}(q)$$
$$= \pi_1^{1*}(\omega_C \otimes \mathcal{L}^{-1}) \otimes \pi_2^{1*} \mathcal{L} \otimes \mathcal{O}_{C^2}(\Delta) | C \times \{q\},$$

and similarly for the restriction to $\{q\} \times C$.

Now suppose $n > 1$ and equation (5) holds for all $n' < n$. Restricting both sides of equation (5) to $C^{2n-2} \times \{p, q\}$ and $\{p_1, q_1, \ldots, p_{n-1}, q_{n-1}\} \times C^2$ for all $p, q, p_1, q_1, \ldots, p_{n-1}, q_{n-1} \in C$, and using the induction hypothesis for $n' = n - 1$ and $n' = 1$ respectively, the Seesaw Principle implies that also equation (5) holds.

STEP III: Consider the embedding $\jmath_p : C \longrightarrow C^{2n}$, $\jmath_p(q) = (p, q, \ldots, p, q)$ and notice that $\delta_{\mathcal{L}}^n \circ \jmath_p = \alpha_{\mathcal{L}, p, -n}$, so that

$$\alpha_{\mathcal{L}, p, -n}^* \mathcal{O}_{J(C)}(\Theta) = \jmath_p^* \delta_{\mathcal{L}}^{n*} \mathcal{O}_{J(C)}(\Theta).$$

It follows that $\alpha_{\mathcal{L}, p, -n}^* \mathcal{O}_{J(C)}(\Theta)$ can be computed from (5). Since

$$(\pi_i^n, \pi_j^n) \circ \jmath_p(q) = \begin{cases} (p, p) & i, j \text{ odd} \\ (q, q) & i, j \text{ even} \\ (p, q) \text{ or } (q, p) & \text{otherwise}, \end{cases}$$

we have that

$$\jmath_p^*(\pi_i^n, \pi_j^n)^* \mathcal{O}_{C^2}(\Delta) = \begin{cases} \mathcal{O}_C & i, j \text{ odd} \\ \omega_C^{-1} & i, j \text{ even} \\ \mathcal{O}_C(p) & \text{otherwise}. \end{cases}$$

It follows that the pull back by $\jmath_p$ of the right hand side of (5) equals $\mathcal{L}^n(n^2 p) \otimes \omega_C^{\frac{n(n-1)}{2}}$. This completes the proof. $\qquad \square$

Proof of Theorem 4.1. By the choice of n we have $h^0(\mathcal{L}(np)) = n$ and the curve $\alpha_{\mathcal{L}, p, -n}(C)$ intersects the divisor Θ properly. The line bundle $(\mathcal{L}(np))^n \otimes \omega_C^{\frac{n(n-1)}{2}}$ has two distinguished divisors, namely $\alpha_{\mathcal{L}, p, -n}^* \Theta$ (according to Proposition 4.2) and $R(\mathcal{L}(np))$ (according to Section 2). Moreover these divisors have the same support, since by definition $q \in \alpha_{\mathcal{L}, p, -n}^* \Theta$ if and only if $\alpha_{\mathcal{L}, p, -n}(q) \in \Theta$, i.e., $h^0(\mathcal{L}(np - nq)) > 0$, and this is the case if and only if q is an inflection point for the line bundle $\mathcal{L}(np)$. For generic $\mathcal{L}$ and p the line bundle $\mathcal{L}(np)$ admits only normal inflection points, so $R(\mathcal{L}(np)) = \sum_{i=1}^{n^2 g} q_i$ with pairwise different points q_i, and hence $R(\mathcal{L}(np)) = \alpha_{\mathcal{L}, p, -n}^* \Theta$. This equality extends to all $\mathcal{L}$ and p for which $\alpha_{\mathcal{L}, p, -n}^* \Theta$ exists and by Lemma 3.1 this is exactly the set $\{(\mathcal{L}, p) \in J(C) \times C \mid h^0(\iota(\mathcal{L})(-np)) = 0\}$. $\qquad \square$

Remark 1. Similarly one can show that if $\alpha_{\mathcal{L}, p, n}(C)$ intersects Θ properly, then $\alpha_{\mathcal{L}, p, n}^* \Theta = R(\iota(\mathcal{L})(np))$.

5. FORMULA(S) FOR THE ORDER OF VANISHING

In this section we prove the following theorem:

Theorem 5.1. *For every* $\mathcal{L} \in \Theta$

$$(6) \qquad \operatorname{ord}_{\mathcal{L}}(\vartheta, X_p) = \sum_m m - g + h^0(\mathcal{L}((g - m)p)),$$

where the sum runs over the g integers m satisfying $h^0(\mathcal{L}((g-m)p)) = h^0(\mathcal{L}((g-m+1)p)) - 1$.

We will obtain it as a direct consequence of the following proposition.

Proposition 5.2. *With n chosen such that $\alpha_{\mathcal{L},p,-n}(C)$ intersects Θ properly (e.g., $n = g$), the order of vanishing of ϑ at $\mathcal{L}$ in the direction X_p is the inflectionary weight of p with respect to $\mathcal{L}(np)$. Therefore,*

$$\text{(7)} \qquad \text{ord}_{\mathcal{L}}(\vartheta, X_p) = \sum_{i=1}^{n} m_i - i.$$

where $\{m_1 < \cdots < m_n\}$ is the gap sequence $G_p(\mathcal{L}(np))$ of $\mathcal{L}(np)$ at p.

Proof. According to Section 3 and Theorem 4.1 we have that

$$\text{ord}_{\mathcal{L}}(\vartheta, X_p) = \text{mult}_p(\alpha^*_{\mathcal{L},p,-n}\Theta) = w_p(\mathcal{L}(np)) = \sum_{i=1}^{n} m_i - i.$$

$\square$

Proof of Theorem 5.1. By definition the sum in equation (6) runs over the set $G_p(\mathcal{L}(gp)) = \{m_1 < \cdots < m_g\}$ of gap numbers of $\mathcal{L}(gp)$ at p. An immediate computation shows that $h^0(\mathcal{L}((g-m_i)p)) = g-i$ for $i = 1, \ldots, g$. So the assertion follows from Proposition 5.2 with $n = g$. $\square$

We now relate Theorem 5.1 to the Formula (1), given by Segal and Wilson. Recall from the introduction the infinite set

$$S_{\mathcal{L}} = \{s \in \mathbb{Z} \mid h^0(\mathcal{L}((s+1)p)) = h^0(\mathcal{L}(sp)) + 1\}.$$

Proposition 5.3. *Denote $S_{\mathcal{L}} = \{s_0 < s_1 < s_2 < \cdots\}$. Then*

$$\text{ord}_{\mathcal{L}}(\vartheta, X_p) = \sum_{i \geq 0} i - s_i.$$

Proof. Note first that $s_0 \geq -\deg\mathcal{L} - 1 = -g$. For $-g \leq s \leq g-1$ we have that $s \in S_{\mathcal{L}}$ if and only if $g - s \in G_p(\mathcal{L}(gp)) = \{m_1 < \cdots < m_g\}$, so that $s_i = g - m_{g-i}$ for $i = 0, \ldots, g-1$. On the other hand $n \in S_{\mathcal{L}}$ for any $n \geq g$ so that $s_n = n$ for any $n \geq g$. Summing up we find

$$\sum_{i \geq 0} i - s_i = \sum_{i=0}^{g-1} i - g + m_{g-i} = \sum_{i=1}^{g} m_i - i.$$

Hence Formula (1) follows from Proposition 5.2. $\square$

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