An interface evolution problem for axisymmetric stressed pore channels

M. Boutat[1], **Y. D'Angelo**[2], **S. Hilout**[3], **V. Lods**[1]

Abstract. The aim of this paper is the mathematical study of the time evolution of a stressed pore channel in an axisymmetric configuration. Under some conditions, morphological instabilities may appear at the material–vacuum interface. Assuming some formal asymptotic assumptions, we derive a nonlinear parabolic PDE (19) governing the cylindrical surface evolution. Local existence and unicity of the solution of this PDE are shown and we also perform some numerical computations (with different parameters and initial condition), using a pseudo–spectral Galerkin method, yielding different behaviours for the solution to (19). In particular, we numerically observe what appears to be a finite time pinch–off.

Keywords. Nonlinear partial differential equations, finite time pinch–off, initial boundary value problem, local solution.

AMS subject classifications. 35K55–35R25–47J35

1 Introduction, presentation of the problem

We consider an axis revolution crystal structure, undergoing stresses with the same symmetry axis. The stress–induced morphological instabilites at the surface of the pore channel [1] are for instance observed when the production process induces a pore distribution that condition the physical and mechanical properties of the materials. Pores have been artificially introduced in titanium ion–implanted sapphire substrates [2] and were observed to heal during high–temperature annealing.

In the mathematical modeling, assuming the load field presents the same axial symmetry as the structure, the full three–dimensional problem (in (r, z, θ) cylindrical coordinates) can be reduced to a 2D problem in (r, z) coordinates. In figure 1 is presented a cylindrical pore of length ℓ and of radius r_0 in a matrix of shear modulus μ and of Poisson coefficient ν. The

[1]Laboratoire de Mathématiques, Université de Poitiers, Boulevard Marie et Pierre Curie, Téléport 2, BP 30179, 86962 Futuroscope Chasseneuil cedex, France, e–mail: mohamed.boutat@math.univ-poitiers.fr (M. Boutat) & vlods@free.fr (V. Lods)

[2]CORIA–INSA, UMR 6614 CNRS, Université de Rouen, 76801 St–Etienne du Rouvray, France, e–mail: dangelo@coria.fr

[3]Département de Mathématiques Appliquées et Informatique, Faculté des Sciences et Techniques, BP 523, 23000 Béni–Mellal, Maroc, e–mail: said_hilout@yahoo.fr

evolution equation for the surface is derived in [1] and reads

$$\frac{\partial r}{\partial \tau} = D(1+r_z^2)^{\frac{1}{2}} \nabla_s^2 \Big(\mathcal{E} + \gamma \mathcal{K} \Big) \quad \text{sur} \quad \Omega_{r(\tau)}, \tag{1}$$

where

• $\Omega_{r(\tau)}$ is the part of the pore which boundary Γ_r is defined by

$$\Gamma_r = \{(r,z);\ r = r(z,\tau)\},$$

The lateral boundaries are given by

$$\Gamma_0 = \{(r,z);\ z = 0\} \quad \text{et} \qquad \Gamma_1 = \{(r,z);\ z = \ell\}.$$

• $r = r(\theta, z, t)$ is the interface radius. It is a function of time t, cylindrical angle θ and of axial coordinate z.

The partial derivative $\dfrac{\partial r}{\partial z}$ will be denoted r_z.

• ∇_s^2 is the surface Laplacian operator [3, 4] defined by :

$$\nabla_s^2 \equiv \frac{1}{q}\left[\frac{\partial}{\partial z}\left(\frac{r^2 + r_\theta^2}{q}\frac{\partial}{\partial z} - \frac{r_z r_\theta}{q}\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{1+r_z^2}{q}\frac{\partial}{\partial \theta} - \frac{r_z r_\theta}{q}\frac{\partial}{\partial z}\right)\right], \tag{2}$$

with $q = \sqrt{r^2(1+r_z^2) + r_\theta^2}$ and $r_\theta = \dfrac{\partial r}{\partial \theta}$.

• $\mathcal{K}$ is the total curvature $\mathcal{K} = \kappa_1 + \kappa_2$ with :

$$\kappa_1 := \frac{1}{r\sqrt{1+r_z^2}}; \quad \kappa_2 := \frac{-r_{zz}}{(1+r_z^2)^{3/2}}. \tag{3}$$

• D is the temperature–dependent diffusion coefficient of the surface atoms.

• γ denotes the free energy of the surface.

• $\mathcal{E}$ is the elastic energy of the structure, defined for all points of $\Omega_{r(\tau)}$.

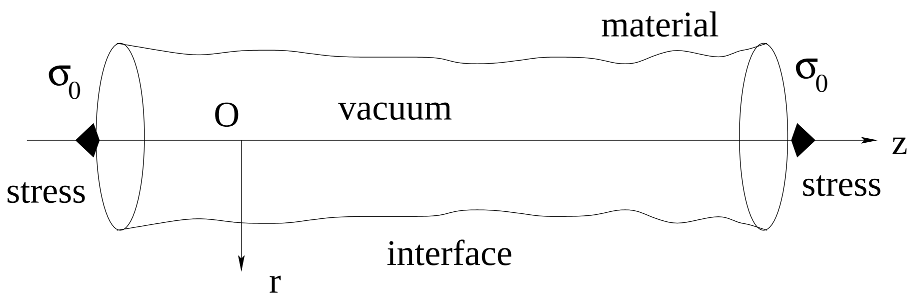


Figure 1: Interface evolution for axisymmetric pore channel under uniaxial constraint σ_0.

A first approach consisted in a theoretical study in absence of constraints [6, 7]. The authors of [6, 7] exploited the results of their experiments, that examined the case of fluids instabilities occuring at the surfaces of cylindrical waterobjects. They demonstrated that instabilities of cylindrical waterobjects occur for fluctuations with wavelenghts higher than the circumference of the cylinder. After these seminal ideas, similar results have been obtained in the case of solids by Nichols and Mullins [8, 4]. They concerned cylindrical trunks (stems). These authors then generalised their results to cylindrical solid pores and precipitates. They were particulary interested by the free–surface morphological evolution of cylindrical objects, by surface or volume diffusion, in the case where the driving force is the surface energy gradient. A computation of the energy variation [5] has also demonstrated that the development of instability is energy–favourable. In [6, 7, 8, 4], the presence of constraints inside the material has not been taken into account. To our knowledge Colin, Grilhé and Junqua [1] have been the first ones to handle the evolution of pores by surface diffusion when the matrix is under constraint. Their treated the case of one, constant, uniaxial constraint σ_0, in the axisymmetrical case. But they also extended the formalism to materials underlying a two–dimensional constraint field, e.g. wiskers and composite fibres; they limited their investigations to structures with cylindrical symmetry [9]. In [1, 10], the crystal structure matrix is loaded with a constant and mono–axial constraint σ_0 and sinusoidal fluctuations are introduced in the lateral surfaces, parallel to Oz. The authors make an energy variation calculation, then a surface kinetic analysis, to caracterize the interface evolution. The lateral surface of the pore is found to be unstable for wavelengths larger than a critical value λ_c that depends on the initial constraint σ_0. For small values of σ_0, the influence on λ_c is weak and the instability is mainly monitored by the surface energy. For large values of σ_0, λ_c becomes very small relative to the pore radius. The growth of the instability becomes much faster. This analysis allows one to determine the control parameters for maintaining the stability of the surface, so that the energy variation remains positive (see [10]). To caracterize Rayleigh instabilities, the authors minimize the energy on the surface and the cylindrical pore evolves to spherical cavities (sockets). The determination of the kinetics of the time evolution of roughnesses by diffusion of pores on surfaces under constraints is also done. Numerical computations show that the new critical wavelenght diminishes when the constraint is higher. In [11, 12], Colin et al. are interested in the shape evolution of cylindrical conducting wires under axial and radial constraints. Instabilities occur simultaneously on the lateral surfaces. The results are similar to those observed during experiments where roughnesses emerge at the interface, on tantale wires in a copper matrix.

High–order perturbation analysis and numerical simulations of the nonlinear equation governing the morphological change of the cylinder surface can be found in [13, 14] and a linear stability analysis in the mono–axial constraint case is done in [15, 9] .

In [8, 4], the authors study the morphological stability of a pore surface when an artificial tension (with zero constraint) is applied. Again, they show the conversion of the cylindrical pores into spheres, and find that the distance between the spheres depends on the surface diffusion and on the bulk diffusion. They also show that the interface becomes unstable if the wavelength becomes larger than the cylinder circumference. In particular, they analyse the external evolution due to the small amplitude perturbations. In the present work, we use the model derived in [1], where the cylinder radius evolution is governed by a parabolic PDE. Under some formal asymptotic assumptions and appropriate scaling laws (to be precised below), this PDE can be further simplified and hence mathematically analysed and numerically solved. The paper is organized as follows. In section 2 is derived the evolution equation, after expanding the elastic energy in terms of the asymptotically small parameter $\alpha \equiv r/\ell$. The proofs of local existence and unicity of the solution to the system (21) are given in section 3. In section 4 are presented some numerical experiments, showing that the solution seems to pinch–off in finite time for some given initial data. Section 5 contains concluding remarks and perspectives.

2 Elastic energy calculation; derivation of the evolution equation

As already mentioned, we use a cylindrical coordinate system (r, z, θ) with Oz as symmetry axis. The axial symmetry allows us to write the radius r as a function of z and τ only : $r = r(z,\tau)$ and $r_\theta = 0$. In order to ensure the cylinder equilibrium, we assume r to be periodic on the boundary faces of $\Omega_r(\tau)$ (Γ_0 and Γ_1). We further assume κ_1 to be negligible in front of κ_2, as in [16]. Equation (1) then reads

$$\frac{\partial r}{\partial \tau} = \frac{D}{r}\frac{\partial}{\partial z}\left[\frac{r}{\sqrt{1+r_z^2}}\frac{\partial}{\partial z}\left(\gamma\frac{-r_{zz}}{(1+r_z^2)^{3/2}} + \mathcal{E}\right)\right]. \tag{4}$$

We also assume that

$$r(z,\tau) \geq r_0 \text{ for all } z \text{ and } \tau \text{ where } r_0 > 0 \text{ is given.} \tag{5}$$

The remaining problem is then to express $\mathcal{E}$ (featured in equation (4)) as a functional of r derivatives.

The elastic energy of the structure can be expressed as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\Big(\sigma(u) - \sigma_0\Big)\Big(\varepsilon(u) - \varepsilon_0\Big) \\ &= \delta\mathcal{E} + \frac{1}{2}\sigma_0\varepsilon_0 \\ &= \frac{1}{2}\lambda \Big(\text{trace}(\varepsilon(u))\Big)^2 + \mu \quad \text{trace}(\varepsilon(u)^2) - \sigma_0\varepsilon(u) + \frac{1}{2}\sigma_0\varepsilon_0 \end{aligned} \tag{6}$$

where ε denotes the linearized deformation tensor

$$\varepsilon(u) \equiv \frac{1}{2}(\nabla u +^t \nabla u) \tag{7}$$

The displacement u verifies the linearized elasticity equations

$$\left\{\begin{array}{rcll} div \quad \sigma(u) & = & 0 & \text{in} \quad \Omega_{r(\tau)} \\ \sigma_F(u).n & = & \sigma_0.n & \text{on} \quad \Gamma_r \\ u/\Gamma_0 & = & u/\Gamma_1 & \end{array}\right. , \tag{8}$$

where $n = (1;0;-r_z)$ is a vector normal to the interface, $\sigma(u)$ is the linearized constraint tensor, σ_0 is the initial constraint tensor (assumed to be a constant). The tangential displacement u_θ is zero since we assume axial symmetry.

After some algebra, and by Hooke's law, system (8) yields a differential system for the two displacements u_r and u_z:

$$\left\{\begin{array}{c} (\lambda+2\mu)\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}u_r\right)+\lambda\frac{\partial^2 u_z}{\partial r\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial r\partial z}+\frac{\partial^2 u_r}{\partial z^2}\right)=0 \\ (\lambda+\mu)\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+\lambda\left(\frac{\partial^2 u_z}{\partial z^2}-\frac{r_z}{r^2}u_r\right)+\mu\left(\frac{\partial^2 u_z}{\partial r^2}+2\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)=0 \\ (\lambda+2\mu)\frac{\partial u_r}{\partial r}+\lambda\left(\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right)-\mu r_z\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)=0 \\ -\lambda r_z\left(\frac{\partial u_r}{\partial r}+\frac{\partial u_z}{\partial z}+\frac{u_r}{r}\right)+\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}-2r_z\frac{\partial u_z}{\partial z}\right)+r_z\sigma_0=0 \end{array}\right. \tag{9}$$

To solve this system (in order to obtain an expression for the elastic energy E), we make an asymptotic expansion of the displacements u_r and u_z. We assume that the small parameter is the ratio of the radius to the length of the pore, i.e.

$$\alpha \equiv r/\ell \ll 1. \tag{10}$$

We then introduce the following scaling laws (change of variables) $R = r$; $Z = \alpha z$ and $t = D\gamma\alpha^4\tau$. Seeking for the displacements u_r and u_z under the forms $u_r(r,z) = U_1(R,Z)$ and $u_z(r,z) = \alpha U_3(R,Z)$, we expand the U_i's in terms of α^2 :

$$U_i = U_i(R;Z;t) = U_i^0 + \alpha^2 U_i^1 + \dots \quad \text{with} \quad i = 1 \text{ and } 3, \tag{11}$$

where the coefficients U_i^j are independent of α. Note that the odd powers of α have zero coefficients.

Introducing the notation $h(t,Z) = r(\tau,z)$, system (9) can be then re–written as

$$\left\{\begin{array}{c} (\lambda+2\mu)\left[\frac{\partial^2 U_1}{\partial R^2}+\frac{1}{R}\frac{\partial U_1}{\partial R}-\frac{1}{R^2}U_1\right]+\alpha^2\left[(\lambda+\mu)\frac{\partial^2 U_3}{\partial R\partial Z}+\mu\frac{\partial^2 U_1}{\partial Z^2}\right]=0 \\ \alpha\left[(\lambda+\mu)(\frac{\partial^2 U_1}{\partial R\partial Z}+\frac{1}{R}\frac{\partial U_1}{\partial Z})-\lambda\frac{h_Z}{R^2}U_1+\mu(\frac{\partial^2 U_3}{\partial R^2}+\frac{1}{R}\frac{\partial U_3}{\partial R})\right]+\alpha^3\left[(\lambda+2\mu)\frac{\partial^2 U_3}{\partial Z^2}\right]=0 \\ \left[(\lambda+2\mu)\frac{\partial U_1}{\partial R}+\lambda\frac{U_1}{R}\right]+\alpha^2\left[\lambda\frac{\partial U_3}{\partial Z}-\mu h_Z(\frac{\partial U_1}{\partial Z}+\frac{\partial U_3}{\partial R})\right]=0 \\ \alpha\left[-\lambda h_Z(\frac{\partial U_1}{\partial R}+\frac{1}{R}U_1)+\mu(\frac{\partial U_3}{\partial R}+\frac{\partial U_1}{\partial Z})+h_Z\sigma_0\right]+\alpha^3\left[-(\lambda+2\mu)h_Z\frac{\partial U_3}{\partial Z}\right]=0 \end{array}\right. \tag{12}$$

By (11)–(12), one obtains a system in the four unknown $U_1^0; U_1^1; U_3^0$ and U_3^1.
Similar calculations (see [17, 16]) yield

$$\begin{aligned} U_1^0 &= 0 \\ U_3^0 &= \frac{-\sigma_0}{\mu} h_Z \ln(R) \quad \text{with} \quad R > 0. \end{aligned} \tag{13}$$

The two coefficients in U_1^1 and U_3^1 appear in the α^4–term of the expansion of $\mathcal{E}$. Hence, we do not need to derive them explicitly.
We then obtain

$$\begin{aligned} \mathcal{E} \;=\; & (\frac{1}{2}\lambda + \mu)\left[(\frac{\partial u_r}{\partial r})^2 + \frac{1}{r^2}u_r^2 + (\frac{\partial u_z}{\partial z})^2\right] + \frac{\mu}{2}(\frac{\partial u_z}{\partial r})^2 + \\ & \frac{\mu}{2}(\frac{\partial u_r}{\partial z})^2 + \frac{\lambda}{r}u_r(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}) + \lambda\frac{\partial u_r}{\partial r}\frac{\partial u_z}{\partial z} + 2\mu\frac{\partial u_r}{\partial z}\frac{\partial u_z}{\partial r} - \sigma_0\frac{\partial u_z}{\partial z} \\ = \; & (\frac{1}{2}\lambda + \mu)\left[(\frac{\partial U_1}{\partial R})^2 + \frac{1}{R^2}U_1^2 + \alpha^4(\frac{\partial U_3}{\partial Z})^2\right] + \frac{\mu}{2}\left[\alpha\frac{\partial U_1}{\partial Z} + \alpha\frac{\partial U_3}{\partial R}\right]^2 + \\ & \lambda\left[\frac{\partial U_1}{\partial R}\frac{U_1}{R} + \alpha^2\frac{\partial U_1}{\partial R}\frac{\partial U_3}{\partial Z} + \alpha^2\frac{U_1}{R}\frac{\partial U_3}{\partial Z}\right] - \alpha^2\sigma_0\frac{\partial U_3}{\partial Z}. \end{aligned} \tag{14}$$

Using (13), the expansion of $\mathcal{E}$ writes

$$\begin{aligned} \mathcal{E} \;=\; & \alpha^2\left[\frac{\mu}{2}(\frac{\partial U_3^0}{\partial R})^2 - \sigma_0\frac{\partial U_3^0}{\partial Z}\right] + \alpha^4\left[\frac{1}{2}(\lambda + 2\mu)(\frac{\partial U_1^1}{\partial R})^2 + \frac{1}{R^2}(U_1^1)^2 + (\frac{\partial U_3^0}{\partial Z})^2 + \right. \\ & \left. \mu\frac{\partial U_3^0}{\partial R}\frac{\partial U_3^1}{\partial R} + \frac{\mu}{2}\frac{\partial U_1^1}{\partial Z}\frac{\partial U_3^0}{\partial R} + \lambda\frac{\partial U_1^1}{\partial R}\frac{U_1^1}{R} + \ldots\right] + \alpha^6\Big[\ldots\Big] \end{aligned} \tag{15}$$

Keeping the α^2–term only yieds

$$\mathcal{E} = \alpha^2\left(\frac{\mu}{2}(\frac{\partial U_3^0}{\partial R})^2 - \sigma_0\frac{\partial U_3^0}{\partial Z}\right)$$

Since

$$\begin{aligned} \frac{\partial U_3^0}{\partial R} &= \frac{-\sigma_0 h_Z}{\mu R} \\ \frac{\partial U_3^0}{\partial Z} &= \frac{-\sigma_0}{\mu} h_{ZZ} \ln(R) \end{aligned}$$

it comes that

$$\mathcal{E} = \alpha^2\frac{\sigma_0^2}{\mu}\left(h_{ZZ}\ln(R) + \frac{1}{2}R^{-2}h_Z^2\right).$$

On the "free" surface $R = h(t, Z)$, we then have

$$\mathcal{E} = \alpha^2\frac{\sigma_0^2}{\mu}\left(h_{ZZ}\ln(h) + \frac{1}{2}h^{-2}h_Z^2\right). \tag{16}$$

Using (16) and introducing $\eta = \dfrac{\sigma_0^2}{\gamma\mu}$, equation (4) becomes :

$$\frac{\partial h}{\partial t} = -\frac{1}{h}\frac{\partial}{\partial Z}\left[h\frac{\partial}{\partial Z}\left((1-\eta\ln(h))h_{ZZ} - \frac{\eta}{2}h^{-2}h_Z^2\right)\right]. \tag{17}$$

To equation (17), we add the two following assumptions

$$r_0 \leq h \qquad \text{and} \qquad h/\ell \ll 1. \tag{18}$$

In the sequel, we shall denote z instead of Z (i.e. $Z \equiv z$). We also make the following change of variable $h(t,z) = e^{(\frac{1}{\eta} + \varphi(t,z))}$. Equation (17) then reads

$$\frac{1}{\eta}\frac{\partial \varphi}{\partial t} = \Lambda(\varphi, \varphi', \varphi'', \varphi^{(3)}, \varphi^{(4)}) \tag{19}$$

with

$$\begin{aligned}\Lambda(\varphi, \varphi', \varphi'', \varphi^{(3)}, \varphi^{(4)}) \;=\; & \varphi\varphi^{(4)} + 2\varphi'\varphi^{(3)} + \varphi''^2 + 8\varphi'^2\varphi'' + 5\varphi\varphi'\varphi^{(3)} + 3\varphi\varphi''^2 + \\ & 9\varphi\varphi'^2\varphi'' + 3\varphi'^4 + 2\varphi\varphi'^4 + e^{-(\frac{1}{\eta}+\varphi)}\left(\varphi'\varphi^{(3)} + \varphi'^2\varphi'' + \varphi''^2\right)\end{aligned} \tag{20}$$

The assumption $r_0 \leq h$ of (18) then writes $\varphi \geq \ln(r_0) - \dfrac{1}{\eta}$. We introduce the quantity r_0 such as $\ln r_0 = \dfrac{1}{\eta}$. Now, φ shoul obey the following PDE problem

$$\left\{\begin{array}{l} \dfrac{1}{\eta}\dfrac{\partial\varphi}{\partial t} = \Lambda(\varphi, \varphi', \varphi'', \varphi^{(3)}, \varphi^{(4)}) \text{ sur }]0,T[\times(0,1) \\ \varphi(t,.) \;\text{ is 1–periodic, given on }\; (0,1) \\ \varphi(0,.) = \varphi_0 \;\text{ is 1–periodic, given on }\; (0,1) \end{array}\right. \tag{21}$$

with $\varphi^{(k)}(t,z) = \dfrac{\partial^k\varphi}{\partial z^k}(t,z)$.

Remark 2.1 *Condition $h/\ell \ll 1$ of (18) is fulfilled as soon as $\ell \gg \varphi + \frac{1}{\eta}$. Let re–write condition (18) under the form*

$$\varphi \geq \ln(r_0) - \frac{1}{\eta} \qquad et \qquad \ell \gg \varphi + \frac{1}{\eta}. \tag{22}$$

In order to study the system (21), we indroduce the periodic Sobolev space

$$H^m_{per}(0,1) = \{f \in H^m(0,1), \quad f^{(i)}(0) = f^{(i)}(1) \text{ for } i = 0,1,\ldots,m-1\}, \tag{23}$$

where $H^m(0,1)$ denotes the usual Sobolev space of index m, for $m \geq 1$. We consider for $t_* > 0$ (t_* will be calculated afterwards) the space $\mathcal{H}$

$$\mathcal{H} = L^2(0,t_*; H^4_{per}(0,1)) \cap L^\infty(0,t_*; H^2_{per}(0,1)).$$

The space $\mathcal{H}$ is endowed with the norm

$$\| \varphi \|_{\mathcal{H}}= \left(\int_0^{t_*} \int_0^1 \varphi^{(4)2}(t,z)\ dz\, dt + \sup_{t\in(0,t_*)} \left[\int_0^1 \varphi''^2(t,z)\ dz + \int_0^1 \varphi^2(t,z)\ dz \right] \right)^{\frac{1}{2}}.$$

We denote by $B_{\mathcal{H}}(0,\xi)$ the closed ball of $\mathcal{H}$ of radius $\xi > 0$.

3 Local existence and uniqueness

In this section we show the local existence and uniqueness of the solution to system (21) in the space $\mathcal{H}$. We make use of the fixed–point Picard theorem [18]. Let Γ be the application of $\mathcal{H}$ in $\mathcal{H}$ defined for any $v \in \mathcal{H}$ by $\Gamma(v) = \varphi$ where φ is solution to the problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Pi(\varphi, \varphi^{(4)}, v, v', v'', v^{(3)}, v^{(4)}) \text{ on }]0,T[\times(0,1) \\ \varphi(t,.) \text{ is a periodic function on } (0,1) \\ \varphi(0,.) = \varphi_0 \text{ is a initial periodic data in } (0,1) \end{cases} \tag{24}$$

where

$$\begin{aligned} \Pi(\varphi, \varphi^{(4)}, v, v', v'', v^{(3)}, v^{(4)}) \quad = \quad & \eta \Big[v\varphi^{(4)} + 2v'v^{(3)} + v''^2 + 8v'^2v'' + 5vv'v^{(3)} + 3vv''^2 + \\ & 9vv'^2v'' + 3v'^4 + 2vv'^4 + e^{-(\frac{1}{\eta} + |v|)} \Big(v'v^{(3)} + v'^2v'' + v''^2 \Big) \Big] \\ & -\varphi^{(4)} + v^{(4)} - \varphi + v \end{aligned} \tag{25}$$

The expression for Π is deduced from that for Λ, defined by (20). It then becomes possible to use fixed–point theorem for Γ. The result of local existence and unicity of the solution to problem (21) is then as follows

Theorem 3.1 *Under condition* (22)*, for any strictly positive initial data* $\varphi_0 \in H^4_{per}(0,1)$ *such that* $\| \varphi_0 \|_{H^4_{per}(0,1)} \leq \xi$*, the problem* (21) *has one local solution* $\Big([0,t_*), \varphi \Big)$ *in* $\mathcal{H}$*.*

In order to show theorem 3.1, we first prove two lemmas. The first one verifies that Γ is well defined under a condition involving ξ, t_* and the initial data φ_0. The second shows that Γ can be a contraction, under an additional condition.

Lemma 3.1 *Let the assumptions of theorem* 3.1 *hold. Then there exists a constant* $c > 0$ *such that for any* (t_*, ξ) $(t_* > 0$ *and* $\xi > 0)$ *satisfying*

$$\begin{cases} 1 - 19\eta(\xi^2 + \xi) - 2\xi \geq \dfrac{1}{2} \\ 4t_*c\eta \Big[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi \Big] + 8t_*\xi + \| \varphi_0 \|^2_{H^2_{per}(0,1)} \leq \xi^2, \end{cases}$$

the application Γ *is well defined to* $B_{\mathcal{H}}(0,\xi)$ *in* $B_{\mathcal{H}}(0,\xi)$*.*

Lemma 3.2 *Let the assumptions of theorem* 3.1 *hold. Then there exists a constant* k $(0 < k < 1)$ *depending on* t_* *and* ξ *such that for all* v_1, v_2 *in* $B_{\mathcal{H}}(0, \xi)$, *the application* Γ *satisfies*

$$\| \Gamma(v_1) - \Gamma(v_2) \| \leq k \| v_1 - v_2 \| .$$

We prove lemmas 3.1 and 3.1 using a priori estimates [19]. The Galerkin method [20] then gives the existence and uniqueness for the solution to system (21).

Proof of lemma 3.1. Let v and φ in $\mathcal{H}$ be such that $\Gamma(v) = \varphi$ then φ satisfies the system

$$\frac{\partial \varphi}{\partial t} = \Pi(\varphi, \varphi^{(4)}, v, v', v'', v^{(3)}, v^{(4)}), \tag{26}$$

i.e.,

$$\begin{aligned} \frac{\partial \varphi}{\partial t} \; = \; & \eta\Big[v\varphi^{(4)} + 2v'v^{(3)} + v''^2 + 8v'^2v'' + 5vv'v^{(3)} + 3vv''^2 + 9vv'^2v'' + 3v'^4 + \\ & 2vv'^4 + e^{-(\frac{1}{\eta} + |v|)} \Big(v'v^{(3)} + v'^2v'' + v''^2 \Big) \Big] - \varphi^{(4)} + v^{(4)} - \varphi + v \end{aligned} \tag{27}$$

Let multiply the two terms of (27) by $\varphi^{(4)}$ and by φ and integrate between 0 and t

$$\begin{aligned} & \int_0^1 \varphi^{(4)} \frac{\partial \varphi}{\partial t} dz + \int_0^1 \varphi^{(4)^2} dz + \int_0^1 \varphi''^2 dz = \\ & \eta\Big[\int_0^1 \varphi^{(4)^2} v dz + 2\int_0^1 \varphi^{(4)} v' v^{(3)} dz + \int_0^1 \varphi^{(4)} v''^2 + 8\int_0^1 \varphi^{(4)} v'^2 v'' dz + \\ & 5\int_0^1 \varphi^{(4)} vv'v^{(3)} dz + 3\int_0^1 \varphi^{(4)} vv''^2 dz + 9\int_0^1 \varphi^{(4)} vv'^2 v'' dz + 3\int_0^1 \varphi^{(4)} v'^4 dz + \\ & 2\int_0^1 \varphi^{(4)} vv'^4 dz + \int_0^1 \varphi^{(4)} e^{-(\frac{1}{\eta} + |v|)} \Big(v'v^{(3)} + v'^2v'' + v''^2 \Big) dz \Big] + \int_0^1 \varphi^{(4)} v^{(4)} dz + \int_0^1 \varphi^{(4)} v dz \\ & = \eta\Big[E_1 + 2E_2 + E_3 + 8E_4 + 5E_5 + 3E_6 + 9E_7 + 3E_8 + 2E_9 + E_{10} + E_{11} + E_{12} \Big] + E_{13} + E_{14} \end{aligned} \tag{28}$$

where

$$E_1 = \int_0^1 \varphi^{(4)^2} v dz; \quad E_2 = \int_0^1 \varphi^{(4)} v' v^{(3)} dz; \quad E_3 = \int_0^1 \varphi^{(4)} v''^2$$

$$E_4 = \int_0^1 \varphi^{(4)} v'^2 v'' dz; \quad E_5 = \int_0^1 \varphi^{(4)} vv'v^{(3)} dz; \quad E_6 = \int_0^1 \varphi^{(4)} vv''^2 dz$$

$$E_7 = \int_0^1 \varphi^{(4)} vv'^2 v'' dz; \quad E_8 = \int_0^1 \varphi^{(4)} v'^4 dz; \quad E_9 = \int_0^1 \varphi^{(4)} vv'^4 dz$$

$$E_{10} = \int_0^1 \varphi^{(4)} e^{-(\frac{1}{\eta} + |v|)} v'v^{(3)} dz; \quad E_{11} = \int_0^1 \varphi^{(4)} e^{-(\frac{1}{\eta} + |v|)} v'^2 v'' dz$$

$$E_{12} = \int_0^1 \varphi^{(4)} e^{-(\frac{1}{\eta} + |v|)} v''^2 dz; \quad E_{13} = \int_0^1 \varphi^{(4)} v^{(4)} dz; \quad E_{14} = \int_0^1 \varphi^{(4)} v dz.$$

Similary

$$\int_0^1 \varphi \frac{\partial \varphi}{\partial t} dz + \int_0^1 \varphi^2 dz + \int_0^1 \varphi''^2 dz = \eta \Big[F_1 + 2F_2 + F_3 + 8F_4 + 5F_5 + 3F_6 + 9F_7 + 3F_8 + 2F_9 + F_{10} + F_{11} + F_{12} \Big] + F_{13} + F_{14} \quad (29)$$

where F_i $(1 \le i \le 14)$ is an expression with the same form as E_i, replacing $\varphi^{(4)}$ by φ.
We use Hölder, Young and interpolation inequalities in the same way as in [16], to estimate the different expressions. We then obtain the following estimations for the right hand side for (28) and (29)

$$\int_0^1 \varphi^{(4)} \frac{\partial \varphi}{\partial t} dz + \int_0^1 \varphi^{(4)2} dz + \int_0^1 \varphi^{''2} dz \le$$

$$(19\eta(\xi + \xi^2) + 2\xi) \int_0^1 \varphi^{(4)2} dz + c\eta \Big[9\xi^9 + 11\xi^8 + 8\xi^7 + 3R^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi \Big] + 2\xi \quad (30)$$

We can re–write (30) as

$$\int_0^1 \varphi^{(4)} \frac{\partial \varphi}{\partial t} dz + (1 - 19\eta(\xi^2 + \xi) - 2\xi) \int_0^1 \varphi^{(4)2} dz + \int_0^1 \varphi^{''2} dz \le$$

$$c\eta \Big[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi \Big] + 2\xi \quad (31)$$

In an analogous way, we can estimate the right hand side of (29) as

$$\int_0^1 \varphi \frac{\partial \varphi}{\partial t} dz + \int_0^1 \varphi^2 dz + \int_0^1 \varphi^{''2} dz \le$$

$$(19\eta(\xi + \xi^2) + 2\xi) \int_0^1 \varphi^2 dz + \eta\xi \int_0^1 \varphi^{(4)2} dz + c\eta \Big[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi \Big] + 2\xi \quad (32)$$

Moreover, we write (32) under the form

$$\int_0^1 \varphi \frac{\partial \varphi}{\partial t} dz + (1 - 19\eta(\xi^2 + \xi) - 2\xi) \int_0^1 \varphi^2 dz + \int_0^1 \varphi^{''2} dz \le$$

$$\eta\xi \int_0^1 \varphi^{(4)2} dz + c\eta \Big[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi \Big] + 2\xi \quad (33)$$

Adding (31) and (33), we obtain

$$\frac{1}{2} \frac{d}{dt} \Big(\int_0^1 \varphi^2 dz + \int_0^1 \varphi''^2 dz \Big) + \int_0^1 \varphi^{(4)2} dz + \int_0^1 \varphi^2 dz + 2 \int_0^1 \varphi^{''2} dz \le$$

$$(19\eta(\xi^2 + \xi) + 2\xi + \eta\xi) \int_0^1 \varphi^{(4)2} dz + (19\eta(\xi^2 + \xi) + 2\xi) \int_0^1 \varphi^2 dz + 2c\eta \Big[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi \Big] + 4\xi \quad (34)$$

Let us then notice that $\int_0^1 \varphi''^2 dz \geq 0$. Then we estimate (34) as

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \varphi^2 dz + \int_0^1 \varphi''^2 dz\right) + (1 - 19\eta(\xi^2+\xi) - 2\xi - \eta\xi)\int_0^1 \varphi^{(4)2} dz +$$
$$+(1 - 19\eta(\xi^2+\xi) - 2\xi)\int_0^1 \varphi^2 dz \leq$$

$$2c\eta\left[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi\right] + 4\xi \tag{35}$$

Let then assume that the following condition (36) holds

$$1 - 19\eta(\xi^2+\xi) - 2\xi \geq \frac{1}{2} \tag{36}$$

Integrating (35) between 0 and t ($0 \leq t \leq t_*$) and using (36) then yield

$$\int_0^1 \varphi^2(t,z)\,dz + \int_0^1 \varphi''^2(t,z)\,dz + \int_0^t\int_0^1 \varphi^{(4)2}(\tau,z)\,dz\,d\tau \leq$$

$$4tc\eta\left[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi\right] + 8t\xi + (\| \varphi_0 \|^2_{L^2(0,1)} + \| \varphi_0'' \|^2_{L^2(0,1)}) \leq$$

$$4t_*c\eta\left[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi\right] + 8t_*\xi + (\| \varphi_0 \|^2_{L^2(0,1)} + \| \varphi_0'' \|^2_{L^2(0,1)}) \tag{37}$$

While passing to the supremum for $t \in (0, t_*)$ on the left hand side of (37), we obtain the following estimate

$\| \varphi \|^2_{\mathcal{H}} = \| \Gamma(v) \|^2_{\mathcal{H}} \leq$

$$4t_*c\eta\left[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi\right] + 8t_*\xi + \| \varphi_0 \|^2_{H^2_{per}(0,1)} \tag{38}$$

We can choose (t_*, ξ) such that $t_* > 0$, $\xi > 0$ and (t_*, ξ) is solution to the following system

$$\begin{cases} 1 - 19\eta(\xi^2+\xi) - 2\xi \geq \frac{1}{2} \\ 4t_*c\eta\left[9\xi^9 + 11\xi^8 + 8\xi^7 + 3\xi^6 + 3\xi^4 + 11\xi^3 + 14\xi^2 + 3\xi\right] + 8t_*\xi + \| \varphi_0 \|^2_{H^2_{per}(0,1)} \leq \xi^2 \end{cases} \tag{39}$$

Since Γ is well defined to $B_{\mathcal{H}}(0,\xi)$ in $B_{\mathcal{H}}(0,\xi)$. □

Remark 3.1 *We use the Faedo–Galerkin method* [20] *with the same a priori estimates than in the proof of lemma* 3.1, *for proving the existence of the solution to system* (24).

Proof of lemma 3.2. Let denote v_1, v_2, φ_1 and φ_2 in $\mathcal{H}$ such that $\Gamma(v_1) = \varphi_1$ and $\Gamma(v_1) = \varphi_2$. Let note $v = v_1 - v_2$ and $\varphi = \varphi_1 - \varphi_2$. φ is solution to system (24), i.e.,

$$\frac{\partial \varphi}{\partial t} = \Pi(\varphi, \varphi^{(4)}, v, v', v'', v^{(3)}, v^{(4)}), \tag{40}$$

or, explicity,

$$\begin{aligned}\frac{\partial\varphi}{\partial t} = {} & \eta\Big[(v_1\varphi_1^{(4)} - v_2\varphi_2^{(4)}) + 2(v_1'v_1^{(3)} - v_2'v_2^{(3)}) + (v_1''^2 - v_2''^2) + 8(v_1'^2v_1'' - v_2'^2v_2'')+ \\ & 5(v_1v_1'v_1^{(3)} - v_2v_2'v_2^{(3)}) + 3(v_1v_1''^2 - v_2v_2''^2) + 9(v_1v_1'^2v_1'' - v_2v_2'^2v_2'') + 3(v_1'^4 - v_2'^4)+ \\ & 2(v_1v_1'^4 - v_2v_2'^4) + e^{-(\frac{1}{\eta}+|v_1|)}\Big(v_1'v_1^{(3)} + v_1'^2v_1'' + v_1''^2\Big)- \\ & e^{-(\frac{1}{\eta}+|v_2|)}\Big(v_2'v_2^{(3)} + v_2'^2v_2'' + v_2''^2\Big)\Big] - \varphi^{(4)} + v^{(4)} - \varphi + v \end{aligned} \tag{41}$$

Let multiply the two sides of (41) respectively by $\varphi^{(4)}$ and φ and integrate between 0 and t we have

$$\begin{aligned} & \int_0^1 \varphi^{(4)}\frac{\partial\varphi}{\partial t}dz + \int_0^1 \varphi^{(4)^2}dz + \int_0^1 \varphi''^2 dz = \\ & \eta\Big[\int_0^1 \varphi^{(4)}(v_1\varphi_1^{(4)} - v_2\varphi_2^{(4)})dz + 2\int_0^1 \varphi^{(4)}(v_1'v_1^{(3)} - v_2'v_2^{(3)})dz + \int_0^1 \varphi^{(4)}(v_1''^2 - v_2''^2)+ \\ & 8\int_0^1 \varphi^{(4)}(v_1'^2v_1'' - v_2'^2v_2'')dz + 5\int_0^1 \varphi^{(4)}(v_1v_1'v_1^{(3)} - v_2v_2'v_2^{(3)})dz+ \\ & 3\int_0^1 \varphi^{(4)}(v_1v_1''^2 - v_2v_2''^2)dz + 9\int_0^1 \varphi^{(4)}(v_1v_1'^2v_1'' - v_2v_2'^2v_2'')dz + 3\int_0^1 \varphi^{(4)}(v_1'^4 - v_2'^4)dz+ \\ & 2\int_0^1 \varphi^{(4)}(v_1v_1'^4 - v_2v_2'^4)dz + \int_0^1 \varphi^{(4)}\Big(e^{-(\frac{1}{\eta}+|v_1|)}v_1'v_1^{(3)} - e^{-(\frac{1}{\eta}+|v_2|)}v_2'v_2^{(3)}\Big)dz+ \\ & \int_0^1 \varphi^{(4)}\Big(e^{-(\frac{1}{\eta}+|v_1|)}v_1'^2v_1'' - e^{-(\frac{1}{\eta}+|v_2|)}v_2'^2v_2''\Big)dz \\ & +\int_0^1 \varphi^{(4)}\Big(e^{-(\frac{1}{\eta}+|v_1|)}v_1''^2 - e^{-(\frac{1}{\eta}+|v_2|)}v_2''^2\Big)dz\Big] + \int_0^1 \varphi^{(4)}v^{(4)}dz + \int_0^1 \varphi^{(4)}v\,dz \\ & = \eta\Big[G_1 + 2G_2 + G_3 + 8G_4 + 5G_5 + 3G_6 + 9G_7 + 3G_8 + 2G_9 + G_{10} + G_{11} + G_{12}\Big] + G_{13} + G_{14} \end{aligned} \tag{42}$$

where

$$G_1 = \int_0^1 \varphi^{(4)}(v_1\varphi_1^{(4)} - v_2\varphi_2^{(4)})dz; \quad G_2 = \int_0^1 \varphi^{(4)}(v_1'v_1^{(3)} - v_2'v_2^{(3)})dz; \quad G_3 = \int_0^1 \varphi^{(4)}(v_1''^2 - v_2''^2)dz$$

$$G_4 = \int_0^1 \varphi^{(4)}(v_1'^2v_1'' - v_2'^2v_2'')dz; \quad G_5 = \int_0^1 \varphi^{(4)}(v_1v_1'v_1^{(3)} - v_2v_2'v_2^{(3)})dz$$

$$G_6 = \int_0^1 \varphi^{(4)}(v_1v_1''^2 - v_2v_2''^2)dz; \quad G_7 = \int_0^1 \varphi^{(4)}(v_1v_1'^2v_1'' - v_2v_2'^2v_2'')dz$$

$$G_8 = \int_0^1 \varphi^{(4)}(v_1'^4 - v_2'^4)dz; \quad G_9 = \int_0^1 \varphi^{(4)}(v_1v_1'^4 - v_2v_2'^4)dz$$

$$G_{10} = \int_0^1 \varphi^{(4)}\Big(e^{-(\frac{1}{\eta}+|v_1|)}v_1'v_1^{(3)} - e^{-(\frac{1}{\eta}+|v_2|)}v_2'v_2^{(3)}\Big)dz$$

$$G_{11}=\int_0^1\varphi^{(4)}\left(e^{-(\frac{1}{\eta}+|v_1|)}v_1'^2v_1''-e^{-(\frac{1}{\eta}+|v_2|)}v_2'^2v_2''\right)dz$$

$$G_{12}=\int_0^1\varphi^{(4)}\left(e^{-(\frac{1}{\eta}+|v_1|)}v_1''^2-e^{-(\frac{1}{\eta}+|v_2|)}v_2''^2\right)dz$$

$$G_{13}=\int_0^1\varphi^{(4)}v^{(4)}dz;\quad G_{14}=\int_0^1\varphi^{(4)}vdz.$$

In an analogous way, we can write

$$\int_0^1\varphi\frac{\partial\varphi}{\partial t}dz+\int_0^1\varphi^2dz+\int_0^1\varphi''^2dz=$$

$$\begin{aligned}&\eta\Big[\int_0^1\varphi(v_1\varphi_1^{(4)}-v_2\varphi_2^{(4)})dz+2\int_0^1\varphi(v_1'v_1^{(3)}-v_2'v_2^{(3)})dz+\int_0^1\varphi(v_1''^2-v_2''^2)+\\&8\int_0^1\varphi(v_1'^2v_1''-v_2'^2v_2'')dz+5\int_0^1\varphi(v_1v_1'v_1^{(3)}-v_2v_2'v_2^{(3)})dz+3\int_0^1\varphi(v_1v_1''^2-v_2v_2''^2)dz+\\&9\int_0^1\varphi(v_1v_1'^2v_1''-v_2v_2'^2v_2'')dz+3\int_0^1\varphi(v_1'^4-v_2'^4)dz+2\int_0^1\varphi(v_1v_1'^4-v_2v_2'^4)dz+\\&\int_0^1\varphi\left(e^{-(\frac{1}{\eta}+|v_1|)}v_1'v_1^{(3)}-e^{-(\frac{1}{\eta}+|v_2|)}v_2'v_2^{(3)}\right)dz+\\&\int_0^1\varphi\left(e^{-(\frac{1}{\eta}+|v_1|)}v_1'^2v_1''-e^{-(\frac{1}{\eta}+|v_2|)}v_2'^2v_2''\right)dz+\\&\int_0^1\varphi\left(e^{-(\frac{1}{\eta}+|v_1|)}v_1''^2-e^{-(\frac{1}{\eta}+|v_2|)}v_2''^2\right)dz\Big]+\int_0^1\varphi v^{(4)}dz+\int_0^1\varphi vdz\\&=\eta\Big[H_1+2H_2+H_3+8H_4+5H_5+3H_6+9H_7+3H_8+2H_9+H_{10}+H_{11}+H_{12}\Big]+H_{13}+H_{14}\end{aligned}\tag{43}$$

where H_i $(1\leq i\leq 14)$ are expressions with the same functionnal form as G_i, by replacing $\varphi^{(4)}$ by φ.

Again, we use the Hölder, Young and interpolation inequalities in the same manner as in [16] to estimate the different expressions. For the term G_{10}, we can write

$$\begin{aligned}G_{10}&=\int_0^1\varphi^{(4)}\left(e^{-(\frac{1}{\eta}+|v_1|)}v_1'v_1^{(3)}-e^{-(\frac{1}{\eta}+|v_2|)}v_2'v_2^{(3)}\right)dz\\&=G_{101}+G_{102}+G_{103}\end{aligned}\tag{44}$$

where

$$G_{101}=\int_0^1\varphi^{(4)}v_1^{(3)}v'e^{-(\frac{1}{\eta}+|v_1|)}dz;\qquad G_{103}=\int_0^1\varphi^{(4)}v_2'v^{(3)}e^{-(\frac{1}{\eta}+|v_2|)}dz$$

$$G_{102}=\int_0^1\varphi^{(4)}v_2'v_1^{(3)}v'\left(e^{-(\frac{1}{\eta}+|v_1|)}-e^{-(\frac{1}{\eta}+|v_2|)}\right)dz$$

We have

$$\begin{aligned}\left| e^{-(\frac{1}{\eta}+|v_1|)} - e^{-(\frac{1}{\eta}+|v_2|)} \right| &= e^{-(\frac{1}{\eta}+|v_2|)} \left| e^{-(|v_1|-|v_2|)} - 1 \right| \\ &\leq e^{|v|} - 1 \\ &= e^{|v|} \left| e^{-|v|} - 1 \right| \leq e^{|v|} |v| \end{aligned} \tag{45}$$

For the exponential terms, we use (45). Since we obtain estimates to expressions (42) and (43) in the following

$$\begin{aligned}&\int_0^1 \varphi^{(4)} \frac{\partial \varphi}{\partial t} dz + \int_0^1 \varphi^{(4)2} dz + \int_0^1 \varphi^{''2} dz \leq (44\eta\xi + 2\xi) \int_0^1 \varphi^{(4)2} dz + \\ &\frac{2}{\xi} \parallel v \parallel_{\mathcal{H}}^2 + \eta \Big[2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + \\ &236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi} \Big] \parallel v \parallel_{\mathcal{H}}^2 \end{aligned} \tag{46}$$

$$\begin{aligned}&\int_0^1 \varphi \frac{\partial \varphi}{\partial t} dz + \int_0^1 \varphi^2 dz + \int_0^1 \varphi^{''2} dz \leq (44\eta\xi + 2\xi) \int_0^1 \varphi^2 dz + \\ &\frac{2}{\xi} \parallel v \parallel_{\mathcal{H}}^2 + \eta \Big[2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + \\ &236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi} \Big] \parallel v \parallel_{\mathcal{H}}^2 \end{aligned} \tag{47}$$

Consequently

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \varphi^2 dz + \int_0^1 \varphi''^2 dz \right) + \int_0^1 \varphi^2 dz + 2\int_0^1 \varphi^{''2} dz + \int_0^1 \varphi^{(4)2} dz \leq \\ &(44\eta\xi + 2\xi) \int_0^1 \varphi^{(4)2} dz + (44\eta\xi + 2\xi) \int_0^1 \varphi^2 dz + \frac{4}{\xi} \parallel v \parallel_{\mathcal{H}}^2 + 2\eta \Big[2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + \\ &236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi} \Big] \parallel v \parallel_{\mathcal{H}}^2 \end{aligned} \tag{48}$$

Since $\displaystyle\int_0^1 \varphi''^2 dz \geq 0$, we can write

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \varphi^2 dz + \int_0^1 \varphi''^2 dz \right) + \Big(1 - (44\eta\xi + 2\xi)\Big) \int_0^1 \varphi^2 dz + \\ &\Big(1 - (44\eta\xi + 2\xi)\Big) \int_0^1 \varphi^{(4)2} dz \leq \Big[\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + \\ &236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi}) \Big] \parallel v \parallel_{\mathcal{H}}^2 \end{aligned} \tag{49}$$

Moreover, let the following technical condition hold

$$1 - (44\eta\xi + 2\xi) \geq \frac{1}{2} \tag{50}$$

Then, we can re–write (49) in the following

$$\frac{1}{2}\frac{d}{dt}\bigg(\int_0^1 \varphi^2 dz + \int_0^1 \varphi''^2 dz\bigg) + \Big(1-(44\eta\xi+2\xi)\Big)\int_0^1 \varphi^{(4)2}dz \leq \bigg[\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + 236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi})\bigg] \parallel v \parallel^2_{\mathcal{H}} \tag{51}$$

Since $\varphi_0 = \varphi_{10} - \varphi_{20} = 0$, and if one integrates (51) between 0 and t, $(0 \leq t \leq t_*)$, one obtains

$$\begin{aligned}\frac{1}{2}\int_0^1 \varphi^2(t,x)\,dz + \frac{1}{2}\int_0^1 \varphi''^2(t,x)\,dz + \Big(1-(44\eta\xi+2\xi)\Big)\int_0^t\int_0^1 \varphi^{(4)2}(\tau,x)\,dz\,d\tau \leq \\ t\bigg[\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + 236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi})\bigg] \parallel v \parallel^2_{\mathcal{H}} \\ \leq t_*\bigg[\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + 236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi})\bigg] \parallel v \parallel^2_{\mathcal{H}}\end{aligned} \tag{52}$$

Condition (50) implies

$$\int_0^1 \varphi^2(t,x)\,dz + \int_0^1 \varphi''^2(t,x)\,dz + \int_0^t\int_0^1 \varphi^{(4)2}(\tau,x)\,dz\,d\tau \leq$$

$$2t_*\bigg[\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + 236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi})\bigg] \parallel v \parallel^2_{\mathcal{H}} \tag{53}$$

Passing to the supremum for $t \in (0, t_*)$ on the left hand side of inequality (53), we obtain the following estimate

$$\int_0^{t_*}\int_0^1 \varphi^{(4)2}(t,x)dzdt + \sup_{t\in(0,t_*)}\bigg(\int_0^1 \varphi^2(t,x)dz + \int_0^1 \varphi''^2(t,x)dz\bigg) \leq$$

$$2t_*\bigg[\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + 236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi})\bigg] \parallel v \parallel^2_{\mathcal{H}} \tag{54}$$

Consequently, Γ is a contraction as soon as the following condition holds true.

$$k = 2t_*\bigg(\frac{4}{\xi} + 2\eta(2\xi + \xi^3 e^{4\xi} + \xi^5 e^{4\xi} + 12\xi c + 236\xi^3 c + 120\xi^5 c + 68\xi^7 c + \xi^3 c e^{4\xi})\bigg) < 1 \tag{55}$$

□

4 Numerical experiments

Since the problem is assumed spatially periodic, we use here a pseudo–spectral method coupled with an exponential scheme that breaks down to a classical forward Euler time scheme, for zero wave number [21, 22]).

In the sequel, we numerically evidence what appears to be a finite time pinch–off of the solution to equation (17)) for some given initial data h_0.

4.1 Spatial discretization

Let consider equation (17) with $h(z,t)$ supposed 2π–periodic. Equation (17) can be written in the form

$$\frac{\partial h}{\partial t} + \mathcal{L}(h) = \mathcal{N}(h) \tag{56}$$

where $\mathcal{L}$ and $\mathcal{N}$ are the linear and non–linear operators of system (17) :

$$\mathcal{L} \equiv \frac{\partial^4}{\partial x^4} \tag{57}$$

and

$$\begin{aligned} \mathcal{N}(h) \quad \equiv \quad & \eta(\ln h)\; h^{(4)} + \eta \Big[h^{-1}(\ln h)\; h' h^{(3)} + 2h^{-1} h' h^{(3)} + h^{-2} h' h^{(3)} - 4h^{-3} h'^2 h'' + h^{-1} h''^2 + \\ & h^{-2} h''^2 + 2h^{-4} h'^4 \Big] - h^{-1} h' h^{(3)} \end{aligned} \tag{58}$$

Periodic boundary conditions and given initial data yield

$$h(0,t) = h(2\pi,t), \quad t \in \mathbb{R}_+,$$

$$h(z,0) = h_0(z), \quad z \in (0,2\pi).$$

The solution to (17) is approximated as a truncated series in the Fourier basis functions $\{(\Phi_k)_{k\in\mathbb{Z}},\, \Phi_k(z) \equiv e^{ikz}\}$:

$$h_N(z,t) = P_N(h(z,t)) = \sum_{k\in\mathbb{I}_N} \hat{h}_k(t)\Phi_k(z),$$

where $\mathbb{I}_N = [1 - \frac{N}{2}, \frac{N}{2}]$; the $\hat{h}_k$ are the spectral coefficients. We require the orthogonality of the residue for all functions of S_N who make up the vectorial space generated by $(\Phi_k)_{k\in\mathbb{Z}}$. In Fourier space, we can write

$$\frac{\partial \hat{h}_k}{\partial t} = \mathcal{L}_k \hat{h}_k + \mathcal{N}_k, \tag{59}$$

where $\mathcal{N}_k$ is the k–th Fourier coefficient of the non–linear term of (56).

4.2 Time discretization

Let $\delta t = t_{n+1} - t_n$ be the (constant) time step size The exponential scheme in time

$$\hat{h}_k^{n+1} = \hat{h}_k^n \; e^{\mathcal{L}_k \delta t} + \mathcal{N}_k \; \frac{e^{\mathcal{L}_k \delta t} - 1}{\mathcal{L}_k} \tag{60}$$

is based on a discrete version of the variable parameter method, that would exactly solve a linear equation. The non–linear term $\mathcal{N}_k$ is computed at each timestep in the direct space, then in the Fourier space, by a discrete fast transform. This first–order in time method can be generalized up to fourth–order, following a Runge–Kutta strategy (ETD or ELP schemes, see e.g. [23, 24, 25]. In the present qualitative study, we limit ourselves to first order. By choosing a relatively small timestep size, we ensure both stability and sufficient precision.
The number N of colocation points was chosen not too large $N = 8192$, typically), to avoid losing precision (and obtaining spurious oscillations) when computing high order derivatives (see the remark about the numerics below). Again, in this qualitative study, no particular strategy was attempted to cure such a (well–known) problem. But an anti–aliasing simple 2/3 rule [26] was applied when computing the non–linear term.

4.3 First numerical test; dissipation

We first tried to solve system (17) for an initial condition inspired from [14] (see also [27]). For the initial condition

$$h_0(z) = 1 + 0.05(\sin(11z) + \sin(10z)) \tag{61}$$

the computed results for $\delta t = 10^{-6}$, $N = 8192$, $\eta = 1$ are represented figures 2, left and right. The observed behaviour, leading to a very quick decay of initial perturbations, is very similar to those of references [14, 27] obtained with a different axially symmetric surface diffusion *constraint–free* modelling.

4.4 Second numerical experiments; pinch–off

After this first experiment, leading to results similar to those of the literature, we tried to solve system (17) for a different, simpler initial condition. For the initial condition

$$h_0(z) = 10 + \sin(z) \tag{62}$$

the computed results for $\delta t = 10^{-3}$, $N = 8192$, $\eta = 0.43$ are represented figures 3, left and right. A zoom around the spike of the solution is represented figure 4, showing how quickly the shape "changes direction".
These results are qualitatively similar to those obtained in the literature. For instance, Sekerka and Marinis [28] model the instability of aligned cylindrical rods in a directionally solidified

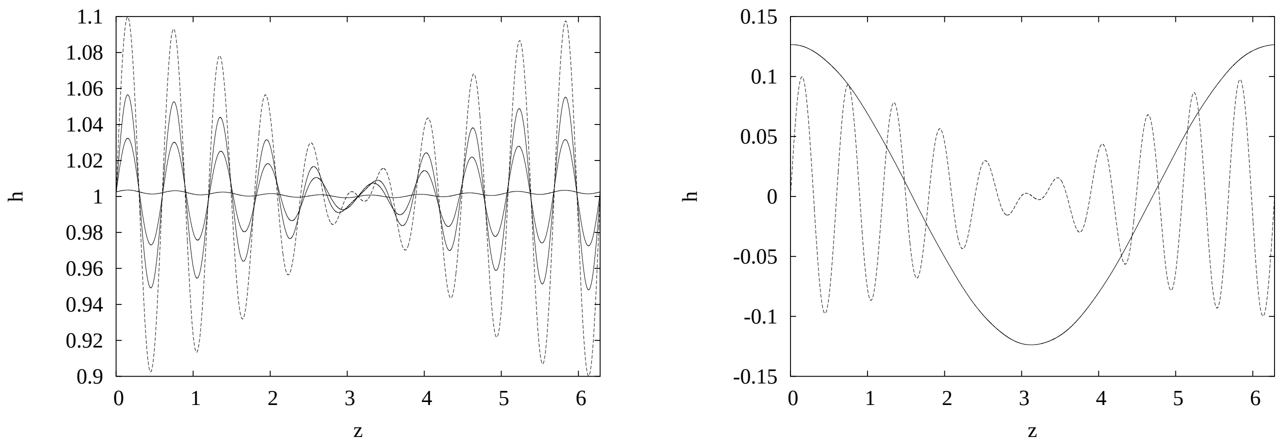


Figure 2: On the left, solution $h(z,t)$ to system (17) with the initial data h_0 (eq. (61)) and for $\eta = 1$; $\delta t = 10^{-6}$ and $N = 8192$. The solution is represented at iterations 0 (initial condition, dotted line), 50, 100 and 400. On the right, solution at times $t = 0$ (dotted line) and $t = 0.001$. To ease readibility, the solutions were translated and the difference between the $t = 0.001$—solution and 1 was scaled by a factor 100.

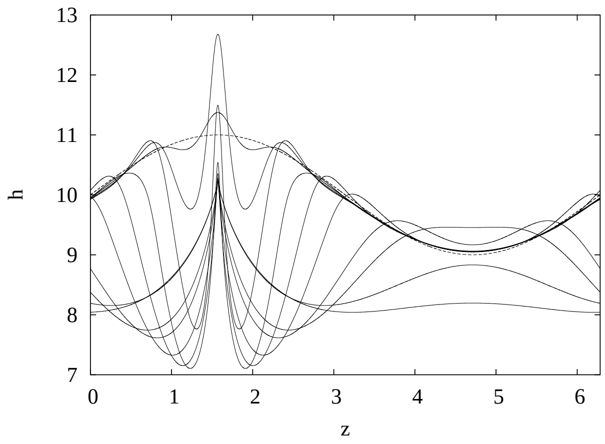
eutectic. They numerically follow the instability until the rods pinch off and begin to coarsen. Coleman, Falk and Moakher [13, 14] showed numerically that Rayleigh unstable cylinders cause a pinching in finite time if perturbed in an axisymmetric way. A result of continuity was established by Bernoff, Bertozzi and Witelski [3] who examined the structure of the pinch–off, showing its self–similar structure. Deckelnick, Dziuk and Elliott [27] were concerned with the analysis of a finite element discretization, based on the above natural splitting of the diffusion fourth–order problem, for axially symmetric surfaces. They show the existence and uniqueness of the discrete solution and also computed some numerical solutions, where the axially symmetric surface diffusion problem also lead to pinch–off.

A remark about the numerics

The pinch–off behaviour leads to an effective discontinuity on the derivatives of the solution. Our spectral approximation can then undergo a Gibbs phenomenon and unwanted oscillations can be obtained in the long–time solution. Classical tentative remedy, like Lanczos–type filtering [29], led to a dissipative behaviour (and a flat shape in the end!). No futher and more elaborate strategy (like Gottlieb et al's Fourier–Gegenbauer reconstruction [30]) was attempted.

5 Concluding remarks

In this work, We consider the case of the axisymmetric profile of a pore, taking into account the effects of elastic strain energy on surface diffusion. Under some formal asymptotic assumptions and scalings, we can obtain a non–linear fourth–order parabolic PDE. We show local existence



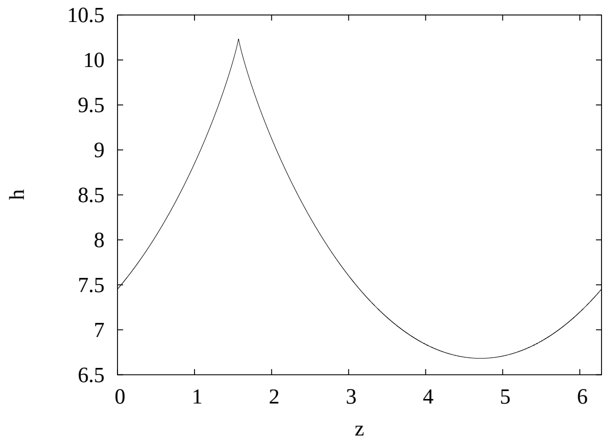


Figure 3: On the left, solution $h(z,t)$ to system (17) with the initial data h_0 (eq. (62)) and for $\eta = 0.43$; $\delta t = 10^{-3}$ and $N = 8192$. The solution is represented at times 0 (initial condition, dotted line), 0.55, 0.6, 0.62, 0.67, 0.75, 0.95, 1.5, 2, 7 and 13. On the right, solution at time $t = 70$. Notice that the long–time behaviour yields a very sharp shape (that strongly resembles a pinch–off) for the solution $h(z,t)$.

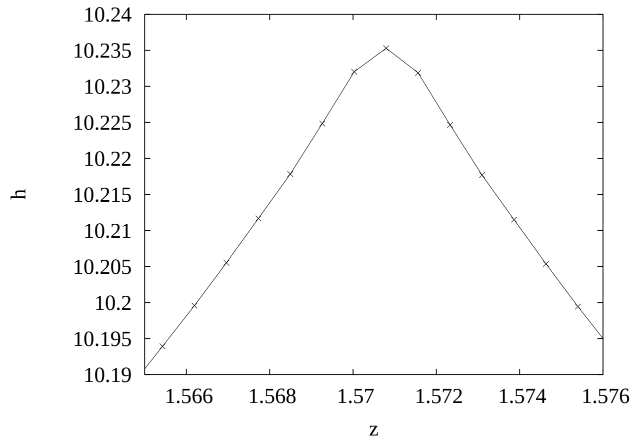


Figure 4: Zoom of the solution $h(z,t)$ to system (17) with the initial data h_0 (eq. (62)) at time $t = 13$ (same run as figure 3). The colocation points are represented as $\times$. The very sharp (mind the scales) change of direction is worth noticing.

and uniqueness to the solution of the problem evolution of the surface of this constrained pore. The shape of cylindrical stressed pore, solution to this PDE (17) depending on a parameter η (proportional to the square of the applied constraint σ_0) is then numerically examined. To approach the solution of the system (17), we adopt a pseudo–spectral method associated with an exponential time scheme. We give some results of the structure of the pinch–off.

Depending on initial condition and on the value of parameter η, the solution can go to a flat shape, or on the contrary, may lead to an apparent discontinuity on the derivative, i.e. a pinch–off behaviour.

These results, leading either to dissipation of initial perturbation or to pinch–off, compare qualitatively well with those of the literature obtained with different modelings of surface diffusion.

Acknowledgements

The authors would like to express their gratitude to Professor J.–M. RAKOTOSON for helpful discussions and remarks.

References

[1] J. Colin, J. Grilhé, N. Junqua, Morphological instabilities of a stressed pore channel, Acta Materialia, 45, (1997), p. 3835–3841.

[2] J.D. Powers, A.M. Glaeser, High–temperature healing of cracklike flaws in titanium ion–implanted sapphire, J. Amer. Ceramic Soc., 76, (1993), p. 2225–2234.

[3] A.J. Bernoff, A.L. Bertozzi, T.P. Witelski, Axisymmetric surface diffusion: dynamics and stability of self–similar pinch–off, J. Stast. Physi., 93, (1998), p. 725–776.

[4] F.A. Nichols, W.W. Mullins, Morphological changes of a surface of revolution due to capillarity–induced surface diffusion, J. of Appl. Phys., 36(6), (1965), p. 1826–1835.

[5] J. Colin, N. Junqua, J. Grilhé, Localized surface instability of non-homogeneously stressed solid, Europhysics. Lett., 38(4), (1997), p. 2609–2621.

[6] M.T. Plateau, Phil. Mag. S4, 12, 286 (1856).

[7] L. Rayleigh, Proc. Lond. Math. Soc., 10, 4 (1879).

[8] W.W. Mullins, Theory of thermal grooving, J. of Appl. Phys., 28(3), (1957), p. 333–339.

[9] D.J. Kirill, S.H. Davis, M.J. Miksis, A.W. Voorhees, Morphological instability of pores and tubules, Interfaces and Free Boundaries, 4, (2002), p. 371–394.

[10] J. Colin, Instabilités de surface de solides contraints, PhD Thesis, Poitiers University, (1997).

[11] J. Colin, Morphological instabilities of stressed axi–symmetrical structures embedded in a matrix: volume diffusion appraoch, Acta Materialia, 52, (2004), p. 4985–4995.

[12] J. Colin, L. Thilly, F. Lecouturier, J.P. Peyrade, S. Askénazy, J. Grilhé, Axial and radial interface instabilities of copper/Tantalum cylindrical conductors, Acta materialia, 47, (1999), p. 2761–2768.

[13] B.D. Coleman, R.S. Falk, M. Moakher, Stability of cylindrical bodies in the theory of surface diffusion, Physica D, 89, (1995), p. 123–135.

[14] B.D. Coleman, R.S. Falk, M. Moakher, Space–time finite element methods for surface diffusion with applications to the theory of the stability of cylinders, SIAM J. Sci. Comp., 17, 6, (1996), p. 1434–1448.

[15] D.J. Kirill, S.H. Davis, M.J. Miksis, A.W. Voorhees, Morphological instability of a whisker, Proc. R. Soc. Lond. A, 455, (1999), p. 3825–3844.

[16] M. Boutat, Y. D'Angelo, S. Hilout, V. Lods, Existence and finite–time blow–up for the solution to a thin–film surface evolution problem, Asymptotic Analysis, 38(2), (2004), p. 93–128.

[17] B.J. Spencer, S.H. Davis, P.W. Woorhees, Morphological instability in epitaxially-strained dislocation-free solid films: nonlinear evolution, Phys. Rev., B, 47, (1993), p. 9760–9777.

[18] H. Brézis, Analyse fonctionnelle, Théorie et applications, Masson, Paris, (1983).

[19] R.A. Adams, Sobolev spaces, Academic Press, New York, (1975).

[20] R. Temam, Infinite–Dimensional Dynamical Systems in Mechanics and Physics, Springer–Verlag, New York, (1988).

[21] Y. D'Angelo, G. Joulin, G. Boury, On model evolution equation for the whole surface of 3D expanding wrinkled premixed flames, Combustion Theory & Modelling, 4, (2000), p. 317–338.

[22] G. Joulin and P. Cambray, On a tentative, approximate evolution equation for markedly wrinkled premixed flames, *Combustion Sci. and Tech.*, 81, (1992), pp. 243–256.

[23] Y. Xu and C.-W. Shu, Local discontinuous Galerkin methods for the Kuramoto-Sivashinsky equations and the Ito–type coupled KdV equations, (submitted to Computer Methods in Applied Mechanics and Engineering, 2005).

[24] G. Beylkin, J.M. Keiser and L. Vozovoi, A new class of time discretization schemes for the solution of nonlinear PDEs, Journal of Computational Physics, Volume 147 , Issue 2 , (1998), pp. 362 – 387.

[25] M. Cox and P. C. Matthews, Exponential time differencing for stiff systems, Journal of Computational Physics, 176 (2002), pp. 430–455.

[26] S.A. Orszag, On the elimination of aliasing in finite–differences schemes by filtering high wavenumbers components, *J. Atmosph. Sc.*, 28, (1971), p. 1074.

[27] K. Deckelnick, G. Dziuk and C.M. Elliott, Error analysis of a semidiscrete numerical scheme for diffusion in axially symmetric surfaces, SIAM J. Num. Anal., 41, 6, (2003), pp. 2161–2179.

[28] R.F. Sekerka, T.F. Marinis, Dynamics of morphological change during solid–solid transformations, Metall. Soc. AIME, (1982), p. 67–84.

[29] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer, (1998).

[30] D. Gottlieb , C.-W. Shu , A. Solomonoff , H. Vandeven, On the Gibbs phenomenon I: recovering exponential accuracy from the Fourier partial sum of a nonperiodic analytic function, Journal of Computational and Applied Mathematics, 43, 1–2, (1992), p. 81–98.