Existence and finite–time blow–up for the solution to a thin–film surface evolution problem

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Abstract. The aim of this paper is to study the evolution of the surface of a crystal structure, constituted by a linearly elastic substrate and a thin film. After appropriate scalings, a formal asymptotical expansion of the displacement, under some asumptions, yields the following nonlinear PDE

$$\frac{\partial h}{\partial t} = -\frac{\partial^2}{\partial x^2}\left((1-\theta h)h'' - \frac{\theta}{2}h'^2\right) \tag{1}$$

where θ is a coefficient related to the crystal, and $h(t,x)$ describes the spatial evolution of the film surface. We give here some results about the finite–time blow–up and prove the existence and uniqueness of a solution in $L^2(0,t_*;H^4_{per}(0,1)) \cap L^\infty(0,t_*;H^2_{per}(0,1))$. We also present some numerical computations confirming the blow–up scenario.

Keywords. Nonlinear partial differential equations, finite time blow-up, initial boundary value problem, local solution.

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1 Introduction

In this work, we consider a constrained crystal structure constituted by an elastic rigid substrate and a free solid film. We are interested in the evolution of the surface of the film in the absence of vapor deposition. This study is related to the modelling of form instabilities for the film, that can be explained by the deformation of the free face as the film thickness exceeds a critical value. This morphological instability is known as Asaro–Tiller–Grinfeld instability [15, 7]. The physical model is developed in [16]. The present mathematical analysis is based on the elasticity equations verified by the solid and on the nonlinear evolution equation describing the film instability (cf. eq. (38)).

In [16], the authors present the details of this model; they analyze the linear stability for the solution to the evolution equation for the surface film in two dimensions of space, and in the

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neighborhood of the critical thickness value. A mathematical analysis of this model is developed in [11, 12]; the authors impose some restrictive assumptions on the model, in the sense that the elastic displacement does not intervene in the evolution equation of the film. The elastic energy hence becomes a function of the free boundary of the solid.

The paper is organized as follows. In section 2 is exposed the mathematical modelling, that in particular introduces some assumptions on the crystal to obtain a formal expansion of the displacement of the elastic structure, after some appropriate scalings. In section 3 is given the proof of the blow–up in finite time for the solution to the problem (38); to this aim, a positive eigenfunction of the membrane problem is made use of. In section 4, we show that (38) admits a local unique solution ; this result is obtained thank to fixed point Picard's theorem. In section 5, we perform some numerical simulations that confirm the above theoretical results.

2 Mathematical modelling

2.1 Basic equations

Let us consider a solid, made up of a film and of an elastic rigid substrate. At time τ, the mesh of this solid occupies an area $\Omega_f(\tau)$ defined by

$$\begin{array}{ccc} f : [0,\infty] \times [0,l_1] \times [0,l_2] & \longrightarrow & I\!R \\ (\tau,x,y) & \longmapsto & f(\tau,x,y) \end{array} \tag{2}$$

and

$$\Omega_f(\tau) = \Big\{(x,y,z), \quad 0 \le x \le l_1; 0 \le y \le l_2; -\infty \le z \le f(\tau,x,y)\Big\} \tag{3}$$

The border between the film and the free surface is denoted by

$$\Gamma_f(\tau) = \{(x,y,z), \quad z = f(\tau,x,y)\} \tag{4}$$

The side surfaces are given by

$$\Gamma_0 = \{(x,y,z), \quad z=0\}, \quad \Gamma_1 = \{(x,y,z), \quad x=0\}, \quad \Gamma_2 = \{(x,y,z), \quad x=\ell_1\} \tag{5}$$

The film is constituted by the points of $\Omega_f(\tau)$ verifying

$$a \le z \le f(\tau,x,y) \tag{6}$$

where a is a strictly positive real, assumed to independent of time. The difference between interatomic distances causes a deformation given by

$$e_0 = \frac{a_F - a_S}{a_S} I \tag{7}$$

where a_F and a_s are interatomic distances related to the film and the substrate respectively. Hooke's law allows us to determine the associated constraints tensor (i.e. the applied stress)

$$\sigma_0 = 2\mu_F \frac{1+\nu_F}{1-2\nu_F} e_0 \tag{8}$$

where μ_F, ν_F are the Lamé and Poisson coefficients of the film at time τ. This tensor generates an elastic displacement u_F and u_S respectively inside the film and the substrate. We assume that the structure is linearly elastic.

This displacement is the solution of the linear elasticity equations

$$\left\{ \begin{array}{ccccc} div \quad \sigma(u) & = & 0 & \text{in} & \Omega_f(\tau) \\ \sigma_F(u_F).n_F & = & \sigma_0.n_F & \text{on} & \Gamma_f(\tau) \\ e(u) & \longrightarrow & 0 & \text{when} & z \to -\infty \\ u_F & = & u_s & \text{on} & \Gamma_a \\ \sigma_F(u_F).n_F & = & \sigma_s(u_s).n_s & \text{on} & \Gamma_a \end{array} \right. \tag{9}$$

with $u = u_F$ in the film and $u = u_s$ in the substrate; n_F and n_s are the external unit normal vectors, to film and substrate respectively; Γ_a denotes the interface film/substrate.

By Hooke's law, the tensor of the constraints in the solid is given by

$$\begin{array}{rcl} \sigma(u) & = & 2\mu\left(\dfrac{\nu}{1-2\nu} Tr(e(u))I + e(u)\right) \\ & = & \lambda Tr(e(u))I + 2\mu e(u) \end{array} \tag{10}$$

where μ is the shear modulus and ν is the Poisson's ratio. The linearized deformation tensor is given by

$$e(u) = \frac{1}{2}\left(\bigtriangledown u + \bigtriangledown u^\top \right) \tag{11}$$

To close the system, one needs a boundary condition on side surfaces. To this aim, we impose the elastic displacement and the amplitude f to be periodic on the sides of $\Omega_f(\tau)$, i.e., on Γ_1 and Γ_2.

We also assume the substrate to be infinite in the z–direction. Then $e(u) \to -\infty$ when $z \to -\infty$ is replaced by $u_s = 0$ on Γ_0. System (9) then becomes

$$\left\{ \begin{array}{ccccc} div \quad \sigma(u) & = & 0 & \text{in} & \Omega_f(\tau) \\ \sigma(u_F).n_F & = & \sigma_0.n_F & \text{on} & \Gamma_f(\tau) \\ u_s & = & 0 & \text{on} & \Gamma_0 \\ u_F & = & u_s & \text{on} & \Gamma_a \\ \sigma_F(u_F).n_F & = & \sigma_s(u_s).n_s & \text{on} & \Gamma_a \\ u/_{\Gamma_1} & = & u/_{\Gamma_2} & & \end{array} \right. \tag{12}$$

According to the model detailed in [16], the evolution of the free face amplitude of displacement f is governed by the equation

$$\frac{\partial f}{\partial \tau} = D(1+ \mid \nabla f \mid^2)^{\frac{1}{2}} \nabla_s^2 \Big(E(u;f) + \gamma K(u;f) \Big) \tag{13}$$

where

$\star$ γ is the surface energy.

$\star$ $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ is the gradient of f towards the space variable (x, y).

$\star$ $\nabla_s f = \dfrac{1}{\Big(1+ \mid \nabla f \mid^2 \Big)^{\frac{1}{2}}} \nabla f$ is the surface gradient and $\nabla_s^2 = \nabla_s . \nabla_s$

$\star$ $E(u;f)$ is the elastic energy of the solid defined in any point of $\Omega_f(\tau)$ given by

$$E(u;f) = \frac{1}{2}\Big(\sigma(u) - \sigma_0\Big).\Big(e(u) - e_0\Big) \tag{14}$$

$\star$ $K(u;f)$ is the reference–state curvature of the film (see [16]). This curvature depends on f and on the displacement $\tilde{u} = u/_{\Gamma_f(\tau)}$ along the evolution surface.

$\star$ D is a diffusion coefficient depending on the temperature.

2.2 Simplifying assumptions

We suppose that the solid constitued by the film and the substrate is infinite in the y–direction. Hence the vector displacement f and the elastic displacement become independent on the y coordinate. The reference–state curvature introduced in [16] is given by

$$K(u;f) = \frac{-1}{H^3}\Big(f''(1 - (\tilde{u}_1)') - (\tilde{u}_3)'' + f'(\tilde{u}_1)'' \Big) \tag{15}$$

where

$$H^2 = 1 + (f')^2 - 2\Big((\tilde{u}_1)' + f'(\tilde{u}_3)' \Big) \tag{16}$$

and

$$\tilde{u}_i\Big(\tau; x\Big) = u_i\Big(\tau; x; f(\tau; x)\Big) \text{ with } i = 1, 2 or 3, \tag{17}$$

$$f' = \frac{\partial f}{\partial x}, \tag{18}$$

$$(\tilde{u}_1)' = \frac{\partial \tilde{u}_1}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial x}.\frac{\partial u_1}{\partial z} = e_{11}(u_F) + f'.\frac{\partial u_1}{\partial z}, \tag{19}$$

$$(\tilde{u}_3)' = \frac{\partial \tilde{u}_3}{\partial x} = \frac{\partial u_3}{\partial x} + f'.\frac{\partial u_3}{\partial z} = \frac{\partial u_3}{\partial x} + f'.e_{33}(u_F). \tag{20}$$

Like in [16], we assume that

$$B = (\tilde{u}_1)' + f'(\tilde{u}_3)' \tag{21}$$

is negligible, compared to $1 + (f')^2$. The curvature $K(u; f)$ can thus be approached by

$$K(f) = \frac{-f''}{H^3} = \frac{-f''}{(1 + (f')^2)^{\frac{3}{2}}} \tag{22}$$

We now have to determine the energy $E(u; f)$. The elastic energy can be written as follows

$$\begin{aligned} E(u; f) &= \frac{1}{2}\Big(\sigma(u) - \sigma_0\Big).\Big(e(u) - e_0\Big) \\ &= \frac{1}{2}\sigma_0.e_0 + \delta E \end{aligned} \tag{23}$$

with

$$\delta E = \frac{1}{2}\lambda\Big(Tr(e(u))\Big)^2 + \mu Tr(e(u)^2) - \sigma_0.e_{11}(u). \tag{24}$$

Thus, we obtain

$$\delta E = \frac{1}{2}\left[\lambda\Big(e_{11}(u) + e_{33}(u)\Big)^2 + 2\mu\Big(e_{11}^2(u) + 2e_{13}^2(u) + e_{33}^2(u)\Big)\right] - \sigma_{011}e_{11} \tag{25}$$

In order to determine δE, one can express $e_{13}(u_F)$; $e_{33}(u_F)$ and $e_{11}(u_F)$ as functions of B. The derivation of the expression of δE is detailed in appendix A. It can be expressed as

$$\delta E = -\frac{1}{2}\frac{\sigma_{011}^2 f'^2(\mu f'^2 + (\lambda + 2\mu))}{\mu(\lambda + 2\mu)\Big(1 + f'^2\Big)^2} - \frac{B\sigma_{011}\mu(\lambda + 2\mu - \lambda f'^2) - 2\mu^2(\lambda + \mu)B^2}{\mu(\lambda + 2\mu)(1 + f'^2)^2} \tag{26}$$

The difficulty now is to calculate B.

2.3 An asymptotic expansion of the energy

The general expression for B, given by equation (21), can be expanded as

$$B = e_{11} + 2f'e_{13} + f'^2 e_{33} \tag{27}$$

where e_{ij} denotes, to simplify, the components of the linearized elasticity tensor $e(u)$. We assume that the film length l_1 is large compared to its height. So, we introduce the following scalings

$$\left\{\begin{array}{rcl} x &=& \frac{1}{\alpha}X \\ z &=& Z \\ f(\tau, x) &=& h(t, X) \\ u_1(t, x, z) &=& \alpha U_1(\tau, X, Z) \\ u_3(t, x, z) &=& U_3(\tau, X, Z) \\ \tau &=& \frac{\alpha^{-4}}{D\gamma}t \end{array}\right. \tag{28}$$

where $\alpha = \frac{1}{l_1}$.
The aim is now to expand the scaled components U_i. After some algebraic manipulations, we can see that we can search an expansion with respect to the powers of α^2 :

$$U_i = U_i^0 + \alpha^2 U_i^1 + \dots \tag{29}$$

Then, it comes

$$\left\{ \begin{array}{lcc} e_{11} & = & \alpha^2 \dfrac{\partial U_1}{\partial X} \\ 2e_{13} & = & \alpha\left(\dfrac{\partial U_1}{\partial Z} + \dfrac{\partial U_3}{\partial X}\right) \\ e_{33} & = & \dfrac{\partial U_3}{\partial Z} \end{array} \right. \tag{30}$$

and we deduce the term B :

$$B = \alpha^2 \frac{\partial U_1}{\partial X} + \alpha f'\left(\frac{\partial U_1}{\partial Z} + \frac{\partial U_3}{\partial X}\right) + f'^2 \frac{\partial U_3}{\partial Z}. \tag{31}$$

In Appendix B are given the details of the procedure that allowed us to compute the asymptotical expansion for the U_i's, B, K' and δE in terms of the small parameter α and of the spatial derivatives $h^{(k)}$ of the scaled displacement amplitude h. Once these are obtained, one can first make use of the following expression for the surface Laplacian in the same terms of α and $h^{(k)}$

$$\begin{array}{rcl} \nabla_s^2 & = & \dfrac{1}{1+f'^2}\left[\dfrac{\partial^2}{\partial x^2}(.) - \dfrac{f'f''}{1+f'^2}\dfrac{\partial}{\partial x}(.)\right] \\ & = & \dfrac{1}{1+\alpha^2 h_X^2}\left[\alpha^2\dfrac{\partial^2}{\partial X^2}(.) - \alpha^3 \dfrac{h_X h_{XX}}{1+\alpha^2 h_X^2}\dfrac{\partial}{\partial X}(.)\right] \end{array} \tag{32}$$

These expression can then be substituted into the governing equation for f

$$\frac{\partial f}{\partial \tau} = D(1+f'^2)^{\frac{1}{2}}\ \nabla_s^2\,(\delta E + \gamma K') \tag{33}$$

to give

$$\alpha^4 D\gamma \frac{\partial h}{\partial t} = D(1+\alpha^2 h_X^2)^{\frac{1}{2}} \frac{1}{1+\alpha^2 h_X^2}\left[\alpha^2 \frac{\partial^2}{\partial X^2}(\delta E + \gamma K') - \frac{\alpha^3 h_X h_{XX}}{1+\alpha^2 h_X^2}\frac{\partial}{\partial X}(\delta E + \gamma K')\right] \tag{34}$$

By replacing δE and K' by their respective values, we obtain

$$\begin{array}{l} \alpha^4 D\gamma \dfrac{\partial h}{\partial t} = D\alpha^2\left[\dfrac{\partial^2}{\partial X^2}\left(\dfrac{\alpha^2\sigma_0^2}{\mu}(hh_{XX} + \dfrac{1}{2}h_X^2) - \alpha^2\gamma h_{XX}\right)\right. \\ \qquad\qquad\qquad \left. -\alpha^3 h_X h_{XX}\dfrac{\partial}{\partial X}\left(\dfrac{\alpha^2\sigma_0^2}{\mu}(hh_{XX} + \dfrac{1}{2}h_X^2) - \alpha^2\gamma h_{XX}\right)\right] \end{array} \tag{35}$$

Since α is assumed to be small, equating the leading terms finally yields

$$\frac{\partial h}{\partial t} \ = \ -\frac{\partial^2}{\partial X^2}\left((1 - \frac{\sigma_0^2}{\gamma\mu}h)h_{XX} - \frac{\sigma_0^2}{2\gamma\mu}h_X^2\right) \tag{36}$$

Let us denote $\theta = \dfrac{\sigma_0^2}{\gamma\mu}$ and x instead of X in the sequel. Then we obtain the evolution equation for the scaled amplitude in its simplified form :

$$\frac{\partial h}{\partial t} = -\frac{\partial^2}{\partial x^2}\left((1-\theta h)h'' - \frac{\theta}{2}h'^2\right) \tag{37}$$

We now consider the problem with initial and boundary conditions

$$\begin{cases} \dfrac{\partial h}{\partial t} = -\dfrac{\partial^2}{\partial x^2}\left((1-\theta h)h'' - \dfrac{\theta}{2}h'^2\right) \text{ in }]0,T[\times(0,1) \\ h(.,t) \text{ is a periodic function on } (0,1), \;\; h^{(k)}(t,0) = h^{(k)}(t,1) = 0 \text{ for } k=0 \text{ and } 2 \\ h(0,.) = h_0 \text{ is a given periodic function on } (0,1). \end{cases} \tag{38}$$

with $h^{(k)}(t,x) = \dfrac{\partial^k h}{\partial x^k}(t,x)$ and $\theta = \sigma_0^2/\mu\gamma$.
Let us now specify the functional framework. Let introduce the space

$$H^m_{per}(0,1) = \{f \in H^m(0,1), \quad f^{(i)}(0) = f^{(i)}(1) \text{ for } i = 0,1,\dots,m-1\} \tag{39}$$

where $H^m(0,1)$ denotes the usual Sobolev space of index m, for $m \geq 1$.
For $t_* > 0$, we consider the space $\mathcal{V}$, defined by

$$\mathcal{V} = L^2(0,t_*;\Omega) \cap L^\infty(0,t_*;H^2_{per}(0,1)), \tag{40}$$

where Ω is a closed space of $H^4_{per}(0,1)$. The space $\mathcal{V}$ is endowed with the norm

$$\| v \|_{\mathcal{V}} = \left(\int_0^{t_*}\int_0^1 v^{(4)2}(t,x)dxdt + \sup_{t\in(0,t_*)}\left(\int_0^1 v''^2(t,x)dx + \int_0^1 v^2(t,x)dx\right)\right)^{\frac{1}{2}}. \tag{41}$$

The symbol $\mathcal{X} = \{v \in \mathcal{V}, \; \| v \|_{\mathcal{V}} \leq R\}$ denotes the closed ball of $\mathcal{V}$ of radius $R > 0$.
Let S be the application of $\mathcal{X}$ in $\mathcal{X}$ such that for all $v \in \mathcal{X}$, S is defined by $Sv = h$ with h solution to the problem, where $\beta > \dfrac{1}{2}$,

$$\begin{cases} \dfrac{\partial h}{\partial t} = -h^{(4)}(1-\theta v) + 3\theta v' v^{(3)} + 2\theta v''^2 - \beta h + \beta v \text{ in }]0,T[\times(0,1) \\ h(t,.) \text{ is a periodic function on } (0,1), \;\; h^{(k)}(t,x) = \dfrac{\partial^k h}{\partial x^k}(t,x) \\ h(0,.) = h_0. \end{cases} \tag{42}$$

Remark 2.1 *By using the change of variable $u = 1 - \theta h$, we can write the thin film equation as*

$$\frac{\partial h}{\partial t} = -\frac{\partial^2}{\partial x^2}\left((1-\theta h)h'' - \frac{\theta}{2}h'^2\right) \tag{43}$$

under the following form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(uu^{(3)} + 2uu''\right) \tag{44}$$

Equation (44) *is hence of the form*

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n u^{(3)} + \alpha u^{n-1} u u'' + \delta u^{n-2} u'^3 \right) \tag{45}$$

with $n = 1$, $\alpha = 2$ *and* $\delta = 0$.
The stationary solutions of (45) *are discussed in [10]. In our case, with our boundary conditions (cf.* (38)*), the unique stationary solution is identically 0.*

3 Blow–up

Many techniques [4] have been used to prove that the solution of a partial differential equation blows up in a finite time (cf. e.g. [17] or [14]). In this section, we make use of the eigenfunction method to investigate the finite time blow–up for the solution to system (38). To this aim, we use the first eigenfunction of the membrane problem. Let introduce the function F, defined on $[0, +\infty[$ for any $t \in [0, +\infty[$ by

$$F(t) = \int_0^1 \varphi_1(x) h(t,x) dx \tag{46}$$

where φ_1 is the first eigenfunction associed to the lowest eigenvalue λ_1 of the membrane problem ([18]), i.e., φ_1 is solution to the eigenvalue problem

$$\left\{ \begin{array}{lll} -\Delta \varphi_1 = \lambda_1 \varphi_1 & \text{in} & \Omega = (0,1) \\ \varphi_1 = 0 & \text{in} & \partial\Omega \\ \varphi_1 > 0 & \text{and} & \displaystyle\int_0^1 \varphi_1(x) dx = 1 \end{array} \right. \tag{47}$$

Theorem 3.1 *Consider the problem* (38)*, with initial condition* $h_0 \in H^4_{per}(0,1)$ $(h_0 > 0)$*, satisfying* $\displaystyle\int_0^1 \varphi_1(x) h_0(x) dx > \frac{2\sqrt{2}}{\theta}$*. If* $h : [0, T_{max}) \times (0,1) \longrightarrow I\!R$ *is a maximal local solution to* (38) *(*T_{max} *can possibly be infinite) with*

$$h \in L^2(0,t; H^4_{per}(0,1)) \cap L^\infty(0,t; H^2_{per}(0,1)), \quad \forall t < T_{max} \tag{48}$$

then h *blows up in finite time, i.e., there exists a time* $t_* < T_{max}$ *such as*

$$\int_0^1 \varphi_1(x) h(t,x) dx \xrightarrow[t \longrightarrow t_*]{} +\infty \tag{49}$$

Proof of theorem 3.1. We can check that F defined by (46) is differentiable and that

$$F'(t) = \int_0^1 \varphi_1(x) \frac{\partial h}{\partial t}(t,x) dx \tag{50}$$

[4]For instance : the concavity method, the comparison method, the logarithmic convexity, the explicit inequality methods...

Since h is solution to (38) and $\varphi_1'(0)-\varphi_1'(1)\geq 0$, by integration by part, we can write (50) under the form

$$\begin{aligned}
F'(t) &\geq -\int_0^1 \varphi_1''(x)\Big((1-\theta h)h''-\frac{\theta}{2}h'^2\Big)dx\\
&= \lambda_1\int_0^1 \varphi_1(x)\Big((1-\theta h)h''-\frac{\theta}{2}h'^2\Big)dx\\
&= -\lambda_1^2\int_0^1 \varphi_1(x)h(t,x)dx+\theta\lambda_1\int_0^1(\varphi_1'hh'+\varphi_1h'^2)dx-\frac{\theta\lambda_1}{2}\int_0^1\varphi_1(x)h'^2(t,x)dx\\
&= -\lambda_1^2F(t)+\frac{\theta\lambda_1^2}{2}\int_0^1\varphi_1(x)h^2(t,x)dx+\frac{\theta\lambda_1}{2}\int_0^1\varphi_1(x)h'^2(t,x)dx
\end{aligned}\tag{51}$$

Since φ_1 is positive and $\lambda_1>0$, it follows that

$$F'(t)\geq -\lambda_1^2F(t)+\frac{\theta\lambda_1^2}{2}\int_0^1\varphi_1(x)h^2(t,x)dx\tag{52}$$

Jensen's inequality hence enables to write

$$\begin{aligned}
\Big(F(t)\Big)^2 &= \Big(\int_0^1\varphi_1(x)h(t,x)dx\Big)^2\\
&\leq \int_0^1\varphi_1^2(x)h^2(t,x)dx\\
&\leq \sqrt{2}\int_0^1\varphi_1(x)h^2(t,x)dx
\end{aligned}\tag{53}$$

Making use of (52) and (53) leads to

$$F'(t)+\lambda_1^2F(t)\geq\frac{\theta\lambda_1^2}{2\sqrt{2}}F^2(t).\tag{54}$$

If we multiply both sides of (54) by $e^{\lambda_1^2t}$ and make use on the identity

$$\frac{d}{dt}\Big(e^{\lambda_1^2t}F(t)\Big)=F'(t)e^{\lambda_1^2t}+\lambda_1^2e^{\lambda_1^2t}F(t)\tag{55}$$

then we obtain the following inequality

$$\frac{d}{dt}\Big(e^{\lambda_1^2t}F(t)\Big)\geq\frac{\theta\lambda_1^2}{2\sqrt{2}}e^{-\lambda_1^2t}\Big(e^{\lambda_1^2t}F(t)\Big)^2.\tag{56}$$

By denoting

$$X(t)=e^{\lambda_1^2t}F(t)\tag{57}$$

equation (56) becomes

$$\frac{d}{dt}X(t)\geq\frac{\theta\lambda_1^2}{2\sqrt{2}}e^{-\lambda_1^2t}X(t)^2\tag{58}$$

that implies that X is monotone increasing. Besides, since $F(0) \geq 0$ by assumption, we then deduce that $X(0)$ and $X(t)$ are positive for all time t. Moreover, by multiplying each sides by X^{-2} and integrating between 0 and t it comes

$$X^{-1}(0) - X^{-1}(t) \geq \frac{\theta}{2\sqrt{2}}(1 - e^{-\lambda_1^2 t}) \tag{59}$$

which can be written as follows

$$X(t) \geq \frac{2\sqrt{2}X(0)}{2\sqrt{2} - \theta X(0)(1 - e^{-\lambda_1^2 t})} \tag{60}$$

Hence, X blows up in a finite time $t_* > 0$ such that

$$2\sqrt{2} - \theta X(0)(1 - e^{-\lambda_1^2 t_*}) = 0 \tag{61}$$

i.e.,

$$t_* = \frac{1}{\lambda_1^2} \ln\left(\frac{\theta X(0)}{\theta X(0) - 2\sqrt{2}}\right) < +\infty \tag{62}$$

under the condition

$$X(0) > \frac{2\sqrt{2}}{\theta} \tag{63}$$

that ends the proof.□

4 A priori estimates

The main result of this section is the following theorem :

Theorem 4.1 *Let* $R = \inf\left(\frac{1}{2(7\theta + \beta)}; \frac{2\beta - 1}{2(6\theta + \beta)}\right)$ *where* $\theta = \sigma_0^2/\mu\gamma$ *and* $\beta > \frac{1}{2}$*. Then, for any initial data* h_0 *in* $H^4_{per}(0,1)$ *such that* $\| h_0 \|_{H^4_{per}(0,1)} < R$*, there exists* $t_* > 0$*, depending on* R*, such that problem* (38) *has one local solution* $\Big([0, t_*), h\Big)$ *with* $h \in L^2(0, t_*; \Omega) \cap L^\infty(0, t_*; H^2_{per}(0,1))$*.*

In order to show theorem 4.1, we made use of fixed point Picard's theorem [2].
Let us first prove two lemmas. The first lemma to be proved verifies that S is well defined under a condition involving R, t_* and the initial data h_0. The second proves that S is a contraction under an additional condition.

Lemma 4.1 *Let suppose that the assumptions of theorem* 4.1 *hold. Then there exists* $t_* > 0$ *such that the application* S *defined by* $Sv = h$ *with* h *solution to problem* (42) *is well defined, in the sense that,* S *is defined for all* $v \in \mathcal{X}$ *and* S *sends* $\mathcal{X}$ *in* $\mathcal{X}$*.*

Proof of lemma 4.1. We here make use of the usual Sobolev's inequalities [1]. In the sequel, c denotes an arbitrary positive constant. Let $v \in \mathcal{X}$ and $Sv = h$ then h satisfies (42), i.e.

$$\frac{\partial h}{\partial t} = -h^{(4)}(1-\theta v) + 3\theta v' v^{(3)} + 2\theta v''^2 - \beta h + \beta v. \tag{64}$$

Let us multiply the two terms of (64) by $h^{(4)}$ and integrate between 0 and 1

$$\begin{aligned}\int_0^1 h^{(4)}\frac{\partial h}{\partial t}dx &= \int_0^1 h^{(4)}\Big(-h^{(4)}(1-\theta v) + 3\theta v' v^{(3)} + 2\theta v''^2 - \beta h + \beta v\Big)dx \\ &= -\int_0^1 h^{(4)2}dx + \theta\int_0^1 h^{(4)2} v dx + 3\theta \int_0^1 h^{(4)} v' v^{(3)} dx + 2\theta\int_0^1 h^{(4)} v''^2 dx \\ &\qquad -\beta\int_0^1 h^{(4)} h dx + \beta \int_0^1 h^{(4)} v dx\end{aligned}$$

Then, it comes

$$\int_0^1 h^{(4)}\frac{\partial h}{\partial t}dx + \int_0^1 h^{(4)2}dx = \theta E_1 + 3\theta E_2 + 2\theta E_3 - \beta E_4 + \beta E_5 \tag{65}$$

where the E_is are defined by

$$\begin{cases} E_1 = \int_0^1 h^{(4)2} v dx, \qquad E_2 = \int_0^1 h^{(4)} v' v^{(3)} dx \\ E_3 = \int_0^1 h^{(4)} v''^2 dx, \qquad E_4 = \int_0^1 h^{(4)} h dx \\ E_5 = \int_0^1 h^{(4)} v dx \end{cases} \tag{66}$$

We then use Hölder and Young inequalities and interpolation in Sobolev spaces to estimate the various expressions.
Since $\mid E_1 \mid \leq \int_0^1 h^{(4)2} \mid v \mid dx \leq \parallel v \parallel_\infty \int_0^1 h^{(4)2} dx$, we have the following inequality

$$\mid E_1 \mid \leq R\int_0^1 h^{(4)2}dx. \tag{67}$$

Applying Hölder and Young inequalities, we obtain

$$\begin{aligned}\mid E_2 \mid &\leq \Big(\int_0^1 h^{(4)2}dx\Big)^{\frac{1}{2}}\Big(\int_0^1 (v' v^{(3)})^2 dx\Big)^{\frac{1}{2}} \\ &\leq R\int_0^1 h^{(4)2}dx + \frac{1}{R}\int_0^1 (v' v^{(3)})^2 dx.\end{aligned} \tag{68}$$

Moreover, since

$$\begin{aligned}\int_0^1 (v' v^{(3)})^2 dx &= \Big(\int_0^1 (v'^6)^{\frac{1}{3}} (v^{(3)3})^{\frac{2}{3}} dx \\ &\leq \Big(\int_0^1 v'^6 dx\Big)^{\frac{1}{3}} \Big(\int_0^1 v^{(3)3} dx\Big)^{\frac{2}{3}} \\ &\leq R\int_0^1 v'^6 dx + \frac{1}{R}\int_0^1 v^{(3)3} dx.\end{aligned} \tag{69}$$

Using the interpolation inequality, we get

$$\int_0^1 v'^6 dx \leq c \parallel v \parallel_{L^2(0,1)}^{\frac{5}{2}} \parallel v'' \parallel_{L^2(0,1)}^{\frac{7}{2}} \tag{70}$$

and

$$\int_0^1 v^{(3)3} dx \leq c \parallel v'' \parallel_{L^2(0,1)}^{\frac{5}{4}} \parallel v^{(4)} \parallel_{L^2(0,1)}^{\frac{7}{4}} \tag{71}$$

Consequently

$$\mid E_2 \mid \leq R \int_0^1 h^{(4)2} dx + \left(c \parallel v \parallel_{L^2(0,1)}^{\frac{5}{2}} \parallel v'' \parallel_{L^2(0,1)}^{\frac{7}{2}} + \frac{c}{R^2} \parallel v'' \parallel_{L^2(0,1)}^{\frac{5}{4}} \parallel v^{(4)} \parallel_{L^2(0,1)}^{\frac{7}{4}} \right). \tag{72}$$

Since $v \in \mathcal{X}$, we deduce that $\parallel v \parallel_{L^2(0,1)}$, $\parallel v'' \parallel_{L^2(0,1)}$ and $\parallel v^{(4)} \parallel_{L^2(0,1)} \leq R$ and consequently, we have the following estimate

$$\mid E_2 \mid \leq R \int_0^1 h^{(4)2} dx + c(R^6 + R) \tag{73}$$

In an analogous way, for E_3, we get

$$\begin{aligned} \mid E_3 \mid \ & \leq \ \left(\int_0^1 h^{(4)2} dx \right)^{\frac{1}{2}} \left(\int_0^1 (v'')^4 dx \right)^{\frac{1}{2}} \\ & \leq \ R \int_0^1 h^{(4)2} dx + \frac{1}{R} \int_0^1 (v'')^4 dx, \end{aligned} \tag{74}$$

and applying the interpolation inequality yields

$$\int_0^1 (v'')^4 dx \leq c \parallel v'' \parallel_{L^2(0,1)}^2 \parallel v^{(4)} \parallel_{L^2(0,1)}^2 \leq cR^4. \tag{75}$$

We hence obtain the following estimate for E_3

$$\mid E_3 \mid \leq R \int_0^1 h^{(4)2} dx + cR^3. \tag{76}$$

Since h is periodic,

$$E_4 = \int_0^1 h^{(4)} h dx = \int_0^1 h''^2 dx. \tag{77}$$

For E_5, we have

$$\begin{aligned} \mid E_5 \mid \ & \leq \ \left(\int_0^1 h^{(4)2} dx \right)^{\frac{1}{2}} \left(\int_0^1 v^2 dx \right)^{\frac{1}{2}} \\ & \leq \ R \int_0^1 h^{(4)2} dx + \frac{1}{R} \int_0^1 v^2 dx \\ & \leq \ R \int_0^1 h^{(4)2} dx + R, \end{aligned} \tag{78}$$

and the following estimate for E_5

$$\mid E_5 \mid \leq R \int_0^1 h^{(4)2} dx + R. \tag{79}$$

By (67)-(73)-(76)-(77)-(79) we have

$$\begin{aligned}\int_0^1 h^{(4)}\frac{\partial h}{\partial t}dx+\int_0^1 h^{(4)2}dx+\beta\int_0^1 h''^2dx &\le \theta R\int_0^1 h^{(4)2}dx+3\theta R\int_0^1 h^{(4)2}dx+3\theta c(R^6+R)\\ &\quad +2\theta R\int_0^1 h^{(4)2}dx+2\theta cR^3+\beta R\int_0^1 h^{(4)2}dx+\beta R\end{aligned}\tag{80}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 h''^2dx+(1-6\theta R-\beta R)\int_0^1 h^{(4)2}dx+\beta\int_0^1 h''^2dx\le\theta c(3R^6+2R^3+3R)+\beta R.\tag{81}$$

Let us multiply the two terms of (64) by h and integrate between 0 and 1, we get

$$\begin{aligned}\int_0^1 h\frac{\partial h}{\partial t}dx &= \int_0^1 h\Big(-h^{(4)}(1-\theta v)+3\theta v'v^{(3)}+2\theta v''^2-\beta h+\beta v\Big)dx\\ &= -\int_0^1 h^{(4)}hdx+\theta\int_0^1 h^{(4)}hvdx+3\theta\int_0^1 hv'v^{(3)}dx+2\theta\int_0^1 hv''^2dx\\ &\qquad -\beta\int_0^1 h^2dx+\beta\int_0^1 hvdx\end{aligned}$$

i.e.

$$\int_0^1 h\frac{\partial h}{\partial t}dx+\int_0^1 h''^2dx+\beta\int_0^1 h^2dx=\theta F_1+3\theta F_2+2\theta F_3+\beta F_4,\tag{82}$$

with

$$\begin{cases}F_1=\displaystyle\int_0^1 h^{(4)}hvdx, & F_2=\displaystyle\int_0^1 hv'v^{(3)}dx\\ F_3=\displaystyle\int_0^1 hv''^2dx, & F_4=\displaystyle\int_0^1 hvdx\end{cases}\tag{83}$$

Once again, Hölder and Young inequalities give

$$\begin{aligned}\mid F_1\mid &\le \Big(\int_0^1 h^{(4)2}dx\Big)^{\frac{1}{2}}\Big(\int_0^1 h^2v^2dx\Big)^{\frac{1}{2}}\\ &\le R\int_0^1 h^{(4)2}dx+\frac{1}{R}\int_0^1 v^2h^2dx,\end{aligned}\tag{84}$$

i.e.,

$$\mid F_1\mid\le R\int_0^1 h^{(4)2}dx+R\int_0^1 h^2dx.\tag{85}$$

As in (73) and (76), we can obtain an estimate for F_2 and F_3

$$\mid F_2\mid\le R\int_0^1 h^2dx+c(R^6+R),\tag{86}$$

and

$$\mid F_3\mid\le R\int_0^1 h^2dx+cR^3.\tag{87}$$

In a similar way, for F_4, we have

$$\begin{aligned}\mid F_4\mid &\le \Big(\int_0^1 h^2dx\Big)^{\frac{1}{2}}\Big(\int_0^1 v^2dx\Big)^{\frac{1}{2}}\\ &\le R\int_0^1 h^2dx+\frac{1}{R}\int_0^1 v^2dx,\end{aligned}\tag{88}$$

i.e.,

$$| F_4 | \leq R \int_0^1 h^2 dx + R. \tag{89}$$

And, by (85)-(86)-(87)-(89), we obtain, substituting in (82)

$$\begin{aligned}\int_0^1 h\frac{\partial h}{\partial t}dx + \int_0^1 h''^2 + \beta\int_0^1 h^2dx \quad \leq \quad & \theta R\int_0^1 h^{(4)2}dx + \theta R\int_0^1 h^2dx + 3\theta R\int_0^1 h^2dx \\ & +3\theta c(R^6+R) + 2\theta R\int_0^1 h^2dx + 2\theta cR^3 \\ & +\beta R\int_0^1 h^2dx + \beta R.\end{aligned} \tag{90}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\int_0^1 h^2dx+\int_0^1 h''^2dx+(\beta-6\theta R-\beta R)\int_0^1 h^2dx-\theta R\int_0^1 h^{(4)2}dx \leq \theta c(3R^6+2R^3+3R)+\beta R. \tag{91}$$

By (81)-(91), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\bigg(\int_0^1 h^2dx + \int_0^1 h''^2dx\bigg) + (\beta-6\theta R-\beta R)\int_0^1 h^2dx \\ &\quad +(1-7\theta R-\beta R)\int_0^1 h^{(4)2}dx + (\beta+1)\int_0^1 h''^2dx \quad \leq \quad 2\theta c(3R^6+2R^3+3R)+2\beta R.\end{aligned} \tag{92}$$

Let us then notice that

$$(\beta+1)\int_0^1 h''^2dx \geq 0 \tag{93}$$

If we integrate (92) between 0 and t, $(0 \leq t \leq t_*)$,we obtain

$$\begin{aligned}&\frac{1}{2}\bigg(\int_0^1 h^2(t,x)dx + \int_0^1 h''^2(t,x)dx\bigg) + (\beta-6\theta R-\beta R)\int_0^t\int_0^1 h^2dxd\tau \\ &+(1-7\theta R-\beta R)\int_0^t\int_0^1 h^{(4)2}dxd\tau \leq 2t\theta c(3R^6+2R^3+3R)+2t\beta R+\frac{1}{2}\bigg(\int_0^1 h_0^2dx+\int_0^1 h_0''^2dx\bigg).\end{aligned} \tag{94}$$

Let us now assume that

$$\begin{cases} 1-7\theta R-\beta R \geq \frac{1}{2} \\ \beta-6\theta R-\beta R \geq \frac{1}{2} \end{cases} \tag{95}$$

Under conditions (95), we deduce from (94) that

$$\begin{aligned}&\int_0^1 h^2(t,x)dx + \int_0^1 h''^2(t,x)dx + \int_0^t\int_0^1 h^{(4)2}(\tau,x)dxd\tau \\ &\leq 4t\theta c(3R^6+2R^3+3R)+4t\beta R+\bigg(\int_0^1 h_0^2dx+\int_0^1 h_0''^2dx\bigg) \\ &\leq 4t_*\theta c(3R^6+2R^3+3R)+4t_*\beta R+\bigg(\int_0^1 h_0^2dx+\int_0^1 h_0''^2dx\bigg).\end{aligned} \tag{96}$$

While passing to the supremum for $t \in (0, t_*)$ on the left hand side of (96) we obtain the following estimate

$$\int_0^{t_*}\int_0^1 h^{(4)2}(t,x)dxdt + \sup_{t\in(0,t_*)}\left(\int_0^1 h^2(t,x)dx + \int_0^1 h''^2(t,x)dx\right) \\ \leq 4t_*\theta c(3R^6+2R^3+3R) + 4t_*\beta R + \| h_0 \|^2_{H^2_{per}(0,1)} \tag{97}$$

i.e.,

$$\| h \|_{\mathcal{X}} = \| Sv \|_{\mathcal{X}} \ \leq \ 4t_*\left(\theta c(3R^6+2R^3+3R) + \beta R\right) + \| h_0 \|^2_{H^2_{per}(0,1)} . \tag{98}$$

Since $h \in \mathcal{X}$, the radius R must be selected such as the following conditions hold

$$\begin{cases} 1 - R(7\theta+\beta) \geq \frac{1}{2} \\ \beta - R(6\theta+\beta) \geq \frac{1}{2} \\ 4t_*\left(\theta c(3R^6+2R^3+3R) + \beta R\right) + \| h_0 \|^2_{H^2_{per}(0,1)} \leq R^2, \end{cases} \tag{99}$$

which are equivalent, for all $\beta > \frac{1}{2}$, to the inequalities $R = \inf\left(\frac{1}{2(7\theta+\beta)}; \frac{2\beta-1}{2(6\theta+\beta)}\right)$ and $4t_*\left(\theta c(3R^6+2R^3+3R)+\beta R\right) \leq R^2 - \| h_0 \|^2_{H^2_{per}(0,1)}$, (since, by assumption, $R > \| h_0 \|_{H^2_{per}(0,1)}$). By using the a priori estimates, the Galerkin's method [18] gives the existence and uniqueness for the solution to system (42), hence S is well defined in $\mathcal{X}$.

Let us now prove the second lemma, stating that S is a contraction.

Lemma 4.2 *Under the condition* (99), *there exists a constant k $(0 < k < 1)$, depending on R and on t_* such that for all v_1 and v_2 in $\mathcal{X}$, we have*

$$\| Sv_1 - Sv_2 \|_{\mathcal{X}} \leq k \| v_1 - v_2 \|_{\mathcal{X}} \tag{100}$$

Proof of lemma 4.2. Let denote $h_1 = Sv_1$ and $h_2 = Sv_2$. Then, h_1 and h_2 are solutions to (12), for $v = v_1$ and $v = v_2$ respectively. Let note

$$h = h_1 - h_2 \quad \text{and} \quad v = v_1 - v_2 \tag{101}$$

By using the same estimates as in lemma 4.1, we have

$$\begin{aligned} \int_0^1 h^{(4)}\frac{\partial h}{\partial t}dx \;=\; & -\int_0^1 h^{(4)2}dx + \theta\int_0^1 h_1^{(4)}h^{(4)}v_1dx - \theta\int_0^1 h_2^{(4)}h^{(4)}v_2dx \\ & +3\theta\int_0^1 h^{(4)}(v_1'v_1^{(3)} - v_2'v_2^{(3)})dx + 2\theta\int_0^1 h^{(4)}v_1''^2dx \\ & -2\theta\int_0^1 h^{(4)}v_2''^2dx - \beta\int_0^1 h^{(4)}hdx + \beta\int_0^1 h^{(4)}vdx, \end{aligned}$$

i.e.,

$$\int_0^1 h^{(4)}\frac{\partial h}{\partial t}dx + \int_0^1 h^{(4)2}dx = \theta G_1 + 3\theta G_2 + 2\theta G_3 + \beta G_4, \tag{102}$$

where

$$\left\{\begin{array}{l} G_1=\int_0^1 h^{(4)}(h_1^{(4)}v_1-h_2^{(4)}v_2)dx, \quad G_2=\int_0^1 h^{(4)}(v_1'v_1^{(3)}-v_2'v_2^{(3)})dx \\ G_3=\int_0^1 h^{(4)}(v_1''^2-v_2''^2)dx, \quad G_4=\int_0^1 h^{(4)}(v-h)dx. \end{array}\right. \tag{103}$$

Moreover, we can write

$$\begin{aligned} \int_0^1 h\frac{\partial h}{\partial t}dx &= -\int_0^1 h''^2dx+\theta\int_0^1 h(h_1^{(4)}v_1-h_2^{(4)}v_2)dx \\ &\quad +3\theta\int_0^1 h(v_1'v_1^{(3)}-v_2'v_2^{(3)})dx+2\theta\int_0^1 h(v_1''^2-v_2''^2)dx \\ &\quad -\beta\int_0^1 h^2dx+\beta\int_0^1 hvdx, \end{aligned} \tag{104}$$

i.e.,

$$\int_0^1 h\frac{\partial h}{\partial t}dx+\int_0^1 h''^2dx+\beta\int_0^1 h^2dx=\theta H_1+3\theta H_2+2\theta H_3+\beta H_4, \tag{105}$$

where

$$\left\{\begin{array}{l} H_1=\int_0^1 h(h_1^{(4)}v_1-h_2^{(4)}v_2)dx, \quad H_2=\int_0^1 h(v_1'v_1^{(3)}-v_2'v_2^{(3)})dx \\ H_3=\int_0^1 h(v_1''^2-v_2''^2)dx, \quad H_4=\int_0^1 hvdx. \end{array}\right. \tag{106}$$

By Hölder and Young inequalities, we have

$$\begin{aligned} \mid G_1\mid &\leq \left(\int_0^1 h^{(4)2}dx\right)^{\frac{1}{2}}\left(\int_0^1 (h_1^{(4)}v_1-h_2^{(4)}v_2)^2dx\right)^{\frac{1}{2}} \\ &\leq R\int_0^1 h^{(4)2}dx+\frac{1}{R}\int_0^1 (h_1^{(4)}v_1-h_2^{(4)}v_2)^2dx. \end{aligned} \tag{107}$$

Since

$$\begin{aligned} \int_0^1 (h_1^{(4)}v_1-h_2^{(4)}v_2)^2dx &= \int_0^1 (h^{(4)}v_1+h_2^{(4)}v)^2dx \\ &\leq 2\int_0^1 h^{(4)2}v_1^2dx+2\int_0^1 h_2^{(4)2}v^2dx \\ &\leq 2R^2\int_0^1 h^{(4)2}dx+2R^2\parallel v\parallel_{\mathcal{X}}^2, \end{aligned} \tag{108}$$

we obtain

$$\mid G_1\mid\leq 3R\int_0^1 h^{(4)2}dx+2R\parallel v\parallel_{\mathcal{X}}^2, \tag{109}$$

and

$$\mid H_1\mid\leq R\int_0^1 h^2dx+2R\int_0^1 h^{(4)2}dx+2R\parallel v\parallel_{\mathcal{X}}^2 . \tag{110}$$

In a similar way, we also have

$$\begin{aligned} \mid G_2\mid &\leq \left(\int_0^1 h^{(4)2}dx\right)^{\frac{1}{2}}\left(\int_0^1 (v_1'v_1^{(3)}-v_2'v_2^{(3)})^2dx\right)^{\frac{1}{2}} \\ &\leq R\int_0^1 h^{(4)2}dx+\frac{1}{R}\int_0^1 (v_1'v_1^{(3)}-v_2'v_2^{(3)})^2dx. \end{aligned} \tag{111}$$

Since we can verify that

$$\begin{aligned} \int_0^1 (v_1' v_1^{(3)} - v_2' v_2^{(3)})^2 dx &= \int_0^1 (v_1' v^{(3)} + v' v_2^{(3)})^2 dx \\ &\leq 2R^2 c \parallel v \parallel_{\mathcal{X}}^2, \end{aligned} \tag{112}$$

we obtain the following estimates for G_2 and H_2

$$\mid G_2 \mid \leq R \int_0^1 h^{(4)2} dx + 2Rc \parallel v \parallel_{\mathcal{X}}^2, \tag{113}$$

$$\mid H_2 \mid \leq R \int_0^1 h^2 dx + 2Rc \parallel v \parallel_{\mathcal{X}}^2 . \tag{114}$$

Moreover, we deduce

$$\begin{aligned} \mid G_3 \mid &\leq \left(\int_0^1 h^{(4)2} dx \right)^{\frac{1}{2}} \left(\int_0^1 v''^4 dx \right)^{\frac{1}{2}} \\ &\leq R \int_0^1 h^{(4)2} dx + \frac{1}{R} \int_0^1 (v_1'' - v_2'')^4 dx. \end{aligned} \tag{115}$$

Since

$$\begin{aligned} \int_0^1 (v_1'' - v_2'')^4 dx = \int_0^1 v''^4 dx &\leq c \parallel v'' \parallel_{L^2(0,1)}^2 \parallel v \parallel_{L^2(0,1)}^2 \\ &\leq cR^2 \parallel v \parallel_{\mathcal{X}}^2 \end{aligned} \tag{116}$$

it comes

$$\mid G_3 \mid \leq R \int_0^1 h^{(4)2} dx + Rc \parallel v \parallel_{\mathcal{X}}^2 . \tag{117}$$

Similarly, we can get the following estimates

$$\begin{aligned} \mid H_3 \mid &\leq \left(\int_0^1 h^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 (v_1''^2 - v_2''^2)^2 dx \right)^{\frac{1}{2}} \\ &\leq R \int_0^1 h^2 dx + \frac{1}{R} \int_0^1 (v_1'' - v_2'')^2 (v_1'' + v_2'')^2 dx \\ &\leq R \int_0^1 h^2 dx + \frac{4R^2}{R} \int_0^1 (v_1'' - v_2'')^2 dx \\ &\leq R \int_0^1 h^2 dx + 4R \int_0^1 v''^2 dx. \end{aligned} \tag{118}$$

The estimate of H_3 is hence given

$$\mid H_3 \mid \leq R \int_0^1 h^2 dx + 4R \parallel v \parallel_{\mathcal{X}}^2 . \tag{119}$$

Since h is periodic,

$$\begin{aligned} G_4 &= \int_0^1 h^{(4)} (v - h) dx = - \int_0^1 h''^2 dx + \int_0^1 h^{(4)} v dx \\ &\leq - \int_0^1 h''^2 dx + R \int_0^1 h^{(4)2} dx + \frac{1}{R} \int_0^1 v^2 dx \\ &\leq - \int_0^1 h''^2 dx + R \int_0^1 h^{(4)2} dx + \frac{1}{R} \parallel v \parallel_{\mathcal{X}}^2 . \end{aligned} \tag{120}$$

Lastly, by using Hölder's and Young's inequalities, we can estimate H_4 :

$$| H_4 | \le R \int_0^1 h^2 dx + \frac{1}{R} \| v \|_{\mathcal{X}}^2 . \tag{121}$$

Using (109)-(113)-(117)-(120) we obtain an estimate for (102)

$$\begin{aligned} \int_0^1 h^{(4)} \frac{\partial h}{\partial t} dx + \int_0^1 h^{(4)2} dx \quad \le \quad & 3\theta R \int_0^1 h^{(4)2} dx + 2\theta R \| v \|_{\mathcal{X}}^2 + 3\theta R \int_0^1 h^{(4)2} dx \\ & + 6\theta Rc \| v \|_{\mathcal{X}}^2 + 2\theta R \int_0^1 h^{(4)2} dx + 2\theta Rc \| v \|_{\mathcal{X}}^2 \\ & - \beta \int_0^1 h''^2 dx + \beta R \int_0^1 h^{(4)2} dx + \frac{\beta}{R} \| v \|_{\mathcal{X}}^2 . \end{aligned} \tag{122}$$

Again (110)-(114)-(119)-(121) yield the following estimate for (105)

$$\begin{aligned} \int_0^1 h \frac{\partial h}{\partial t} dx + \int_0^1 h''^2 dx + \beta \int_0^1 h^2 dx \quad \le \quad & \theta R \int_0^1 h^2 dx + 2\theta R \int_0^1 h^{(4)2} dx + 2\theta R \| v \|_{\mathcal{X}}^2 \\ & + 3\theta R \int_0^1 h^2 dx + 6\theta R \| v \|_{\mathcal{X}}^2 + 2\theta R \int_0^1 h^2 dx \\ & + 8\theta R \| v \|_{\mathcal{X}}^2 + \beta R \int_0^1 h^2 dx + \frac{\beta}{R} \| v \|_{\mathcal{X}}^2 . \end{aligned} \tag{123}$$

Consequently, (122) and (123) give

$$\frac{1}{2} \frac{d}{dt} \Big(\int_0^1 h^2 dx + \int_0^1 h''^2 dx \Big) + \int_0^1 h^{(4)2} dx + \int_0^1 h''^2 dx + \beta \int_0^1 h^2 dx \le \tag{124}$$

$$R(10\theta + \beta) \int_0^1 h^{(4)2} dx + (12\theta R + 14\theta c R + \frac{2\beta}{R}) \| v \|_{\mathcal{X}}^2 + (6\theta R + \beta R) \int_0^1 h^2 dx - \beta \int_0^1 h''^2 dx. \tag{125}$$

Let us notice that

$$(\beta + 1) \int_0^1 h''^2 dx \ge 0 \tag{126}$$

thus we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \Big(\int_0^1 h^2 dx + \int_0^1 h''^2 dx \Big) + \Big(\beta - R(6\theta + \beta) \Big) \int_0^1 h^2 dx & \\ + \Big(1 - R(10\theta + \beta) \Big) \int_0^1 h^{(4)2} dx \quad & \le \quad \Big(\theta R(12 + 14c) + \frac{2\beta}{R} \Big) \| v \|_{\mathcal{X}}^2 . \end{aligned} \tag{127}$$

According to condition (99), we have

$$\beta - R(6\theta + \beta) \ge \frac{1}{2} \tag{128}$$

thus we can simplify expression (127)

$$\frac{1}{2} \frac{d}{dt} \Big(\int_0^1 h^2 dx + \int_0^1 h''^2 dx \Big) + \Big(1 - R(10\theta + \beta) \Big) \int_0^1 h^{(4)2} dx \le \Big(\theta R(12 + 14c) + \frac{2\beta}{R} \Big) \| v \|_{\mathcal{X}}^2 . \tag{129}$$

We integrate (129) between 0 and t, $(0 \leq t \leq t_*)$ and we know that $h_0 = h_{10} - h_{20} = 0$, then

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 h^2(t,x)dx + \frac{1}{2}\int_0^1 h''^2(t,x)dx \\
&+\Big(1 - R(10\theta+\beta)\Big)\int_0^t\int_0^1 h^{(4)2}(\tau,x)dxd\tau \;\leq\; \Big(\theta R(12+14c) + \frac{2\beta}{R}\Big) \parallel v \parallel_{\mathcal{X}}^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq\; t_*\Big(\theta R(12+14c) + \frac{2\beta}{R}\Big) \parallel v \parallel_{\mathcal{X}}^2 .
\end{aligned}
\tag{130}
$$

If we assume that

$$1 - R(10\theta+\beta) \geq \frac{1}{2} \tag{131}$$

(130) becomes

$$\int_0^1 h^2(t,x)dx + \int_0^1 h''^2(t,x)dx + \int_0^t\int_0^1 h^{(4)2}(\tau,x)dxd\tau \leq 2t_*\Big(\theta R(12+14c)+\frac{2\beta}{R}\Big) \parallel v \parallel_{\mathcal{X}}^2, \tag{132}$$

while passing to the supremum for $t \in (0,t_*)$ on the left hand side of (132), we have the following estimate

$$\int_0^{t_*}\int_0^1 h^{(4)2}(t,x)dxdt + \sup_{t\in(0,t_*)}\Big(\int_0^1 h^2(t,x)dx + \int_0^1 h''^2(t,x)dx\Big) \leq$$

$$2t_*\Big(\theta R(12+14c) + \frac{2\beta}{R}\Big) \parallel v \parallel_{\mathcal{X}}^2 . \tag{133}$$

Consequently, S is a contraction as soon as the following condition is satisfied

$$k = 2t_*\Big(\theta R(12+14c) + \frac{2\beta}{R}\Big) < 1. \tag{134}$$

Let us now make the following remarks :

Remark 4.1 *

1. *S is well defined and contracting if conditions* (99)-(131)-(134) *hold, i.e.,*

$$
\begin{cases}
1 - R(10\theta+\beta) \geq \dfrac{1}{2} \\
\beta - R(6\theta+\beta) \geq \dfrac{1}{2} \\
4t_*\Big(\theta c(3R^6 + 2R^3 + 3R) + \beta R\Big) + \parallel h_0 \parallel_{H^2_{per}(0,1)}^2 \leq R^2 \\
2t_*\Big(\theta R(12+14c) + \dfrac{2\beta}{R}\Big) < 1.
\end{cases}
\tag{135}
$$

2. *If condition* (135) *holds, then the fixed point Picard's theorem* [2] *allows us to conclude that there exists an unique solution h to* (38) *with $h \in L^2(0,t_*;\Omega) \cap L^\infty(0,t_*;H^2_{per}(0,1))$.*

Corollary 4.1 *There exists a time $t'_* > 0$, depending on R and on initial data h_0 such as if $h = h_0 + \delta h$ is a local solution to* (38) *with*

$$h \in L^2(0, t'_*; H^4_{per}(0,1)) \cap L^\infty(0, t'_*; H^2_{per}(0,1)). \tag{136}$$

Then δh is a local solution to the problem

$$\begin{cases} \dfrac{\partial g}{\partial t} = -\dfrac{\partial^2}{\partial x^2}\left((1-\theta g)g'' - \dfrac{\theta}{2}g'^2\right) + \theta\dfrac{\partial^2}{\partial x^2}\left(h_0 g'' + h_0'' g + h_0' g'\right) + f \text{ in }]0,T[\times]0,1[\\ g(t,.) \text{ is a periodic function on } (0,1) \\ g(0,.) = 0 \end{cases} \tag{137}$$

with

$$\delta h \in L^2(0, t'_*; H^4_{per}(0,1)) \cap L^\infty(0, t'_*; H^2_{per}(0,1)) \tag{138}$$

and where

$$f = -\frac{\partial^2}{\partial x^2}\left((1-\theta h_0)h_0'' - \frac{\theta}{2}h_0'^2\right). \tag{139}$$

Proof of corollary 4.1. The proof is the same as that of theorem 4.1. It is sufficient to consider the application ϕ defined in $\mathcal{X}$ on $\mathcal{X}$ for any $v \in \mathcal{X}$ by $\phi v = g$ where g is solution of problem

$$\begin{cases} \dfrac{\partial g}{\partial t} = -g^{(4)}(1-\theta v) + 3\theta v' v^{(3)} + 2\theta v''^2 - \beta g + \beta v + 4\theta h_0'' v'' + 3\theta h_0' v^{(3)} + 3\theta h_0^{(3)} v' \\ \qquad\qquad +\theta h_0 v^{(4)} + \theta h_0^{(4)} v + f \quad \text{ in } \quad]0,T[\times]0,1[\\ g(.,t) \text{ is a periodic function on } (0,1) \\ h(0,.) = 0, \end{cases} \tag{140}$$

where $\beta > \dfrac{1}{2}$. We use the same a priori estimates of lemmas 4.1 and 4.2 for the term

$$-g^{(4)}(1-\theta v) + 3\theta v' v^{(3)} + 2\theta v''^2 - \beta g + \beta v + f, \tag{141}$$

and the Hölder and Young inequalities and interpolation in Sobolev spaces for the term

$$4\theta h_0'' v'' + 3\theta h_0' v^{(3)} + 3\theta h_0^{(3)} v' + \theta h_0 v^{(4)} + \theta h_0^{(4)} v. \tag{142}$$

5 Some numerical experiments

In order to validate the above theoretical results, we numerically solved the system (38) with different initial conditions h_0. Since the problem is periodic in space, we adopted a pseudo–spectral method coupled with an exponential scheme in time [8, 19]. The nonlinear partial differential equation verified by the scaled amplitude $h(x,t)$ is of the form

$$\frac{\partial h}{\partial t} = \mathcal{L}(h) + \mathcal{N}(h) \tag{143}$$

where $\mathcal{L}$ and $\mathcal{N}$ represent linear and nonlinear spatial operators, respectively. If we expand the solution $h(x,t)$ to (38) in the Fourier space[5]

$$h(x,t) = \sum_{k \geq 0} \hat{h}_k(t) \exp(kx) \tag{144}$$

where $\hat{h}_k(t) \in \mathbb{C}$ are the Fourier coefficients of $h(x,t)$. Equation (143) hence becomes

$$\frac{\partial \hat{h}_k}{\partial t} = L_k \times \hat{h}_k + N_k \tag{145}$$

The above expression is then advanced in time, from time n to $n+1$. Denoting δt the timestep size, we have

$$\hat{h}_k^{n+1} = \hat{h}_k^n \exp(L_k \delta t) + N_k^n (\exp(L_k \delta t) - 1)/L_k \tag{146}$$

This temporal scheme is based on a discrete version of the variable–parameter method, that would solve exactly a linear equation. The nonlinear term N_k is computed at each timestep in the direct space, then in the Fourier space, by a discrete fast transform. We choose to use a constant δt and equidistributed collocation points. For the computation of the nonlinear term, we applied the simple and popular Orszag 2/3 de–aliasing rule [13, 4].

5.1 Preliminary benchmark

In order to validate the numerical procedure, we first successfully applied it to a benchmark where an analytical solution can be found. Namely, we numerically solved the viscous Burgers'equation [3], that contains a linear (viscous) and a nonlinear (quadratic) flux term :

$$u_t + \frac{1}{2}(u^2)_x = u_{xx} \tag{147}$$

and where $u_0(x) \equiv u(x, t=0)$ is given. It is then well known that a Hopf–Cole transformation [5, 9] $u = -2v_x/v$ yields a heat equation for v.

5.2 First test–case

We first tried to solve system (38) for an initial $h_0(x) \equiv h(x, t=0)$ chosen equal to

$$h_0(x) = \alpha_0(\sin(2x) + M) \tag{148}$$

with $\alpha_0 = 0.4$. The M additive coefficient in equation (148) is aimed to insure the pointwise positivity of the initial condition h_0. It is chosen equal to $M \equiv 1.1$. We shall see in the sequel that this condition is not crucial and will not change the qualitative behaviour of the solution. For the boundary conditions, we only impose periodicity on $[-\pi;\pi]$ (instead of (0;1),

[5]For the sake of simplicity, we assumed that the solution $h(x,t)$ is 2π–periodic in space, instead of 1–periodic.

for simplicity) and we did not take into account the (somewhat restrictive) zero condition on the second derivative. Moreover, following condition (63), we numerically determined (thanks to a dichotomy procedure) the critical value θ_{crit} of the parameter θ featured in system (38). For $\theta \geq \theta_{\text{crit}}$, the solution $h(x,t)$ blows up in finite time. For $\theta < \theta_{\text{crit}}$, the solution shrinks to the mean value $< h_0 >$ of the initial condition $h_0(x)$.

Actually, the precise numerical computed value for the critical coefficient θ_{crit} is not only depending on h_0 as expected, but was also found to be slightly dependent on the number of collocation points and even (but less sensitively) on the timestep size. After some preliminary trials in the different cases, and also for the sake of consistency of all the numerical results, we choose the number of collocation points for all our numerical computations to be 8192. The timestep size is also chosen constant and identical for all the numerical tests, fixed to $\delta t = 10^{-3}$. These values allowed to keep the above numerical artefact negligible and gave sufficiently accurate results for all the treated cases, at a reasonable CPU cost. Figure 1 shows the early time evolution of the solution $h(x,t)$. All the computations for this case (figures 1 to 5) were performed with a numerical value of $\theta = 1.728825$, very close to the critical value. Indeed, for $\theta = 1.728824$, the behaviour changes drastically and the solution shrinks to the (approximate) mean value of the initial condition.

5.3 Second test–case

After this first round of calculations, we then tried to avoid pure sinusoids as initial conditions. Since the pseudo–spectral method have spectral precision when handling C^∞ functions, we still imposed the initial condition to be smooth, but this time with a rather "shaked" pattern, shown in figure 6.

This function is arbitrarily given by

$$h_0^r(x) = \alpha_0^r(\sin\left[x - \sin 8x + \sin 4(x+c_1) + 2\sin 3(x+c_2)\right] + 1/3\sin 17x + 2/3\sin 7x + M^r) \tag{149}$$

with $c_1 = -2.7657654675$ and $c_2 = 1.8754858580$ chosen "randomly". The value of the constant M^r can be chosen such as $h_0^r(x)$ be positive, or such as the mean value of $h_0^r(x)$ be zero (and hence lose positivity for the initial condition, see section 5.1).

In this section, we shall only consider the positive case, and choose $M^r = 2.4$ to insure it. For $\alpha_0^r = 0.4$, the value of θ is fixed to 0.776218262, just above its critical value (for $\theta = 0.775913239$, we observed no blowing). Again, the number of collocation points is still equal to 8192, and the timestep size is again $\delta t = 10^{-3}$.

Sample results can be found on figures 7 and 8. The qualitative behaviour is quite similar to the pure sinusoidal case.

Moreover, despite the visual spikes (around $x \simeq \pm 2.1$ of figure 8), the solution remains smooth.

Remark 5.1 *In the pure sinusoidal case, we also tried to run some computations without the positivity condition, i.e. for* $M = 0$ *and* $h_0(x) \equiv \alpha_0 \sin(2x)$ *(see section* 5.2*). Qualitatively, there still exists a critical value* θ_{crit} *such as the solution blows up for* $\theta \geq \theta_{crit}$*. This critical value seems to be (exactly!) related to the amplitude* α_0 *of the initial pattern. The relation is astonishingly as simple as*

$$\alpha_0 \theta_{crit} = constant \equiv \theta_{crit}(\alpha_0 = 1) \equiv K \tag{150}$$

Notice that for the positive case (equation (148) *for* $M = 1.1$*), the seemingly exact relationship* 150 *is also valid. Interestingly, we observed the same behaviour (i.e. that relationship* (150) *still holds) even in the case of an initial "shaky" pattern of the kind given by equation* (149)*.*

6 Concluding Remarks

In this paper, we hence prove the existence, unicity and finite–time blow–up for the solution to a problem describing the evolution of the free surface of a film (problem (38)). The numerical findings are coherent with the theoretical results, and even suggest that our (technical) mathematical assumptions might be weakened, like the positivity condition for the initial pattern, or the zero boundary condition for the second space derivative of h. We confirmed the existence of a critical value of the parameter θ appearing in problem (38) : for θ above this critical value θ_{crit} , the solution actually blows up in finite time; for $\theta < \theta_{\text{crit}}$, the solution shkinks to the mean value of the initail pattern. The numerical computations also suggest that this critical value may be very simply linked to the amplitude of the initial pattern (see (150)).
For this modelling, further investigations may be envisaged : i) directly computing the displacement of the free surface by use of a finite element method; ii) modifying the modelling to allow multi–valuedness for the function describing the free surface. To accomplish ii), a preliminary step may consist in studying the axisymmetric case, like in [6].

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Appendix

A Derivation of the expression for the energy δE

The boundary condition on $\Gamma_f(\tau)$ can be written as follows

$$\sigma(u_F).n_F = \sigma_0.n_F \tag{151}$$

where

$$n_F = \frac{1}{\sqrt{1+f'^2}}\Big(-f'; 1\Big) \tag{152}$$

and

$$\sigma(u_F) = \lambda Tr\Big(e(u)\Big)I + 2\mu e(u). \tag{153}$$

Since only the component σ_{011} of the tensor σ_0 is not equal to zero byassumption, we deduce the following equations

$$\left\{\begin{array}{rcr} -\lambda f' e_{33} + 2\mu e_{13} & = & -\sigma_{011} f' + (\lambda+2\mu) f' e_{11} \\ (\lambda+2\mu)e_{33} - 2\mu f' e_{13} & = & -\lambda e_{11}. \end{array}\right. \tag{154}$$

Consequently, we obtain $e_{13}(u)$ and $e_{33}(u)$ as functions of $e_{11}(u)$, f and B on $\Gamma_f(\tau)$.

$$\left\{\begin{array}{rcl} e_{13} & = & \dfrac{1}{2}\dfrac{-\sigma_{011}\dfrac{\lambda+2\mu}{\mu}f' + 4(\lambda+\mu)f' e_{11}}{(\lambda+2\mu)-\lambda f'^2} \\ e_{33} & = & -\dfrac{\sigma_{011} f'^2 + \Big(\lambda - (\lambda+2\mu)f'^2\Big)e_{11}}{(\lambda+2\mu)-\lambda f'^2} \end{array}\right. \tag{155}$$

We now have to determine $e_{11}(u_F)$ as a function of B. Let us recall that B is given as follows

$$B = (\tilde{u}_1)' + f'(\tilde{u}_3)', \tag{156}$$

where

$$\tilde{u}\Big(\tau, x\Big) = u\Big(\tau, x, f(\tau,x)\Big), \tag{157}$$

and

$$\frac{\partial \tilde{u}_1}{\partial x} = (\tilde{u}_1)' = \frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial x}.\frac{\partial u_1}{\partial z} = e_{11}(u_F) + f'.\frac{\partial u_1}{\partial z}. \tag{158}$$

Moreover, we can write

$$\frac{\partial u_1}{\partial z} = 2e_{13} - \frac{\partial u_3}{\partial x} \tag{159}$$

and

$$(\tilde{u}_3)' = \frac{\partial u_3}{\partial x} + f' e_{33}, \tag{160}$$

which leads to the equality

$$B = (\tilde{u}_1)' + f'(\tilde{u}_3)' = e_{11} + f'\left(2e_{13} - \frac{\partial u_3}{\partial x}\right) + f'\frac{\partial u_3}{\partial x} + f'^2 e_{33} = e_{11} + 2f'e_{13} + f'^2 e_{33} \quad (161)$$

hence

$$e_{11} = B - 2f'e_{13} - f'^2 e_{33}. \quad (162)$$

By subsituting e_{13} and e_{33} with their respective expressions, we obtain

$$e_{11}(u_F) = \frac{\sigma_{011} f'^2\left(\lambda + 2\mu + \mu f'^2\right) + \mu\left(\lambda + 2\mu - \lambda f'^2\right)B}{\mu\left(\lambda + 2\mu\right)\left(1 + f'^2\right)^2}. \quad (163)$$

Consequently, we can calculate the energy as a function of B

$$\begin{aligned}
\delta E &= \frac{1}{2}\left[\lambda\left(e_{11} + e_{33}\right)^2 + 2\mu\left(e_{11}^2 + 2e_{13}^2 + e_{33}^2\right)\right] - \sigma_{011} e_{11} \\
&= -\frac{1}{2}\frac{\left(\sigma_{011} f'^2 + 2\mu B\right)\left(f'^2\mu\sigma_{011} + \sigma_{011}(\lambda + 2\mu) - 2B\mu(\lambda + \mu)\right)}{\mu\left(\lambda + 2\mu\right)\left(1 + f'^2\right)^2} \\
&= -\frac{1}{2}\frac{\sigma_{011}^2 f'^2(\mu f'^2 + (\lambda + 2\mu))}{\mu(\lambda + 2\mu)\left(1 + f'^2\right)^2} - \frac{B\sigma_{011}\mu(\lambda + 2\mu - \lambda f'^2) - 2\mu^2(\lambda + \mu)B^2}{\mu(\lambda + 2\mu)(1 + f'^2)^2}.
\end{aligned} \quad (164)$$

B Derivation of the asymptotic expressions for the U_i's, B, K' and δE

Hooke's law can be written as follows, by using scalings (30),

$$\left\{\begin{aligned}
\sigma_{11} &= \alpha^2\left(\lambda + 2\mu\right)\frac{\partial U_1}{\partial X} + \lambda\frac{\partial U_3}{\partial Z} \\
\sigma_{13} &= \mu\alpha\left(\frac{\partial U_1}{\partial Z} + \frac{\partial U_3}{\partial X}\right) \\
\sigma_{33} &= \alpha^2\lambda\frac{\partial U_1}{\partial X} + \left(\lambda + 2\mu\right)\frac{\partial U_3}{\partial Z}.
\end{aligned}\right. \quad (165)$$

The partial derivatives of the σ_{ij} are then given by

$$\left\{\begin{aligned}
\partial_1\sigma_{11} &= \alpha\lambda\frac{\partial^2 U_3}{\partial X\partial Z} + \alpha^3\left(\lambda + 2\mu\right)\frac{\partial^2 U_1}{\partial X^2} \\
\partial_1\sigma_{13} &= \mu\alpha^2\left(\frac{\partial^2 U_1}{\partial X\partial Z} + \frac{\partial^2 U_3}{\partial X^2}\right) \\
\partial_3\sigma_{13} &= \alpha\mu\left(\frac{\partial^2 U_1}{\partial Z^2} + \frac{\partial^2 U_3}{\partial X\partial Z}\right) \\
\partial_3\sigma_{33} &= \alpha^2\lambda\frac{\partial^2 U_1}{\partial X\partial Z} + \left(\lambda + 2\mu\right)\frac{\partial^2 U_3}{\partial Z^2}.
\end{aligned}\right. \quad (166)$$

Thus, the linearized elasticity equations

$$\left\{\begin{array}{rclll} div \quad \sigma(u) & = & 0 & \text{in} & \Omega_f(\tau) \\ \sigma_F(u_F).n_F & = & \sigma_0.n_F & \text{on} & \Gamma_f(\tau) \\ u & = & 0 & \text{on} & \Gamma_0 \end{array}\right. \tag{167}$$

can be expanded as

$$\left\{\begin{array}{rclll} \partial_1\sigma_{11} + \partial_3\sigma_{13} & = & 0 & \text{in} & \Omega_f(\tau) \\ \partial_1\sigma_{13} + \partial_3\sigma_{33} & = & 0 & \text{in} & \Omega_f(\tau) \\ -\sigma_{11}f' + \sigma_{13} & = & -\sigma_0 f' & \text{on} & \Gamma_f(\tau) \\ -\sigma_{13}f' + \sigma_{33} & = & 0 & \text{on} & \Gamma_f(\tau). \end{array}\right. \tag{168}$$

By using (166), (168) and the equality $f' = \alpha h_X$, it comes

$$\left\{\begin{array}{lll} 0 = \alpha\Big[\mu\dfrac{\partial^2 U_1}{\partial Z^2} + \Big(\lambda+\mu\Big)\dfrac{\partial^2 U_3}{\partial X\partial Z}\Big] + \alpha^3\Big[\Big(\lambda+2\mu\Big)\dfrac{\partial^2 U_1}{\partial X^2}\Big] & \text{in} & \Omega_f(\tau) \\ 0 = \Big[\Big(\lambda+2\mu\Big)\dfrac{\partial^2 U_3}{\partial Z^2}\Big] + \alpha^2\Big[\mu\dfrac{\partial^2 U_3}{\partial X^2} + \Big(\lambda+\mu\Big)\dfrac{\partial^2 U_1}{\partial X\partial Z}\Big] & \text{in} & \Omega_f(\tau) \\ 0 = \alpha\Big[-\lambda h_X\dfrac{\partial U_3}{\partial Z} + \mu\Big(\dfrac{\partial U_1}{\partial Z} + \dfrac{\partial U_3}{\partial X}\Big) + \sigma_0 h_X\Big] + \alpha^3\Big[-h_X\Big(\lambda+2\mu\Big)\dfrac{\partial U_1}{\partial X}\Big] & \text{on} & \Gamma_f(\tau) \\ 0 = \Big[\Big(\lambda+2\mu\Big)\dfrac{\partial U_3}{\partial Z}\Big] + \alpha^2\Big[\lambda\dfrac{\partial U_1}{\partial X} - \mu h_X\Big(\dfrac{\partial U_1}{\partial Z} + \dfrac{\partial U_3}{\partial X}\Big)\Big] & \text{on} & \Gamma_f(\tau) \\ \alpha U_1 = U_3 = 0 & \text{on} & \Gamma_0. \end{array}\right. \tag{169}$$

We then use the formal expansions

$$\begin{array}{rcl} U_1 & = & U_1^0 + \alpha^2 U_1^1 + \ldots \\ U_3 & = & U_3^0 + \alpha^2 U_3^1 + \ldots . \end{array} \tag{170}$$

The equtions (169), (170) allow to calculate

$$U_1^0; U_1^1; U_3^0; U_3^1. \tag{171}$$

For instance, U_3^0 satisfies the system

$$\left\{\begin{array}{rclll} \Big(\lambda+2\mu\Big)\dfrac{\partial^2 U_3^0}{\partial Z^2} & = & 0 & \text{in} & \Omega_f(\tau) \\ \Big(\lambda+2\mu\Big)\dfrac{\partial U_3^0}{\partial Z} & = & 0 & \text{on} & \Gamma_f(\tau) \\ U_3^0 & = & 0 & \text{on} & \Gamma_0, \end{array}\right. \tag{172}$$

and U_1^0 solves the equations

$$\left\{\begin{array}{rcl}\mu\dfrac{\partial^2 U_1^0}{\partial Z^2}+\Big(\lambda+\mu\Big)\dfrac{\partial^2 U_3^0}{\partial Z\partial X} & = & 0 \ \text{ in } \ \Omega_f(\tau)\\ -\lambda h_X\dfrac{\partial U_3^0}{\partial Z}+\mu\Big(\dfrac{\partial U_1^0}{\partial Z}+\dfrac{\partial U_3^0}{\partial X}\Big)+\sigma_0 h_X & = & 0 \ \text{ on } \ \Gamma_f(\tau)\\ U_1^0 & = & 0 \ \text{ on } \ \Gamma_0.\end{array}\right. \tag{173}$$

We can then determine U_3^1 (with the order α^2)

$$\left\{\begin{array}{rcl}\Big(\lambda+2\mu\Big)\dfrac{\partial^2 U_3^1}{\partial Z^2}+\mu\dfrac{\partial^2 U_3^0}{\partial X^2}+\Big(\lambda+\mu\Big)\dfrac{\partial^2 U_1^0}{\partial X\partial Z} & = & 0 \ \text{ in } \ \Omega_f(\tau)\\ \Big(\lambda+2\mu\Big)\dfrac{\partial U_3^1}{\partial Z}+\lambda\dfrac{\partial U_1^0}{\partial X}-\mu h_X\Big(\dfrac{\partial U_1^0}{\partial Z}+\dfrac{\partial U_3^0}{\partial X}\Big) & = & 0 \ \text{ on } \ \Gamma_f(\tau)\\ U_3^1 & = & 0 \ \text{ on } \ \Gamma_0,\end{array}\right. \tag{174}$$

and, U_1^1 (with the order α^3) is solution of the system

$$\left\{\begin{array}{rcl}\Big(\lambda+2\mu\Big)\dfrac{\partial^2 U_1^0}{\partial X^2}+\mu\dfrac{\partial^2 U_1^1}{\partial Z^2}+\Big(\lambda+\mu\Big)\dfrac{\partial^2 U_3^1}{\partial X\partial Z} & = & 0 \ \text{ in } \ \Omega_f(\tau)\\ -h_X\Big(\lambda+2\mu\Big)\dfrac{\partial U_1^0}{\partial X}+\mu\Big(\dfrac{\partial U_1^1}{\partial Z}+\dfrac{\partial U_3^1}{\partial X}\Big)-\lambda h_X\dfrac{\partial U_3^1}{\partial Z} & = & 0 \ \text{ on } \ \Gamma_f(\tau)\\ U_1^1 & = & 0 \ \text{ on } \ \Gamma_0.\end{array}\right. \tag{175}$$

Thus, we can deduce that

$$\begin{array}{rcl}U_3^0 & = & 0\\ U_1^0 & = & -\dfrac{\sigma_0}{\mu}h_X Z\\ U_3^1 & = & \dfrac{1}{2}\Big(\dfrac{\lambda+\mu}{\lambda+2\mu}\Big)\dfrac{\sigma_0}{\mu}h_{XX}Z^2-\dfrac{\sigma_0}{\lambda+2\mu}\Big(hh_{XX}+h_X^2\Big)Z\\ U_3^1 & = & ZR+\dfrac{1}{2}Z^2S\\ U_1^1 & = & ZT+\dfrac{1}{2}UZ^2+\dfrac{1}{6}VZ^3\end{array} \tag{176}$$

with

$$\begin{aligned}
R &= \frac{-\sigma_0}{\Big(\lambda+2\mu\Big)}\Big(hh_{XX}+h_X^2\Big) \\
S &= \Big(\frac{\lambda+\mu}{\lambda+2\mu}\Big)\frac{\sigma_0}{\mu}h_{XX} \\
T &= \frac{-\sigma_0(3\lambda+2\mu+2)}{2\mu(\lambda+2\mu)}h^2h_{XXX}-\frac{\sigma_0(4\lambda+3\mu+3)}{\mu(\lambda+2\mu)}hh_Xh_{XX}-\frac{\lambda\sigma_0}{\mu(\lambda+2\mu)}h_X^3 \\
U &= \frac{\sigma_0}{\mu\Big(\lambda+2\mu\Big)}\Big(3h_Xh_{XX}+hh_{XXX}\Big) \\
V &= \sigma_0\frac{\Big(2\lambda+3\mu\Big)}{\mu\Big(\lambda+2\mu\Big)}h_{XXX}.
\end{aligned} \tag{177}$$

We finally obtain the expansions

$$U_1 = U_1^0+\alpha^2U_1^1+\dots = \frac{-\sigma_0}{\mu}h_XZ+\alpha^2\Big(ZT+\frac{1}{2}UZ^2+\frac{1}{6}VZ^3\Big)+\dots \tag{178}$$

$$U_3 = U_3^0+\alpha^2U_3^1+\dots = \alpha^2\Big(ZR+\frac{1}{2}Z^2S\Big)+\dots \tag{179}$$

By neglecting the term in α^4, we deduce that

$$\begin{aligned}
B &= \alpha^2\frac{\partial U_1}{\partial X}+\alpha^2h_X\Big(\frac{\partial U_1}{\partial Z}+\frac{\partial U_3}{\partial X}\Big)+\alpha^2h_X^2\frac{\partial U_3}{\partial Z} \\
&= \alpha^2\Big(\frac{\partial U_1}{\partial X}+h_X\frac{\partial U_1}{\partial Z}+h_X\frac{\partial U_3}{\partial X}+h_X^2\frac{\partial U_3}{\partial Z}\Big) \\
&= \alpha^2\Big(\frac{\partial U_1^0}{\partial X}+\alpha^2\frac{\partial U_1^1}{\partial X}+h_X\frac{\partial U_1^0}{\partial Z}+\alpha^2h_X\frac{\partial U_1^1}{\partial Z}+h_X\frac{\partial U_3^0}{\partial X}+\alpha^2h_X\frac{\partial U_3^1}{\partial X}+h_X^2\frac{\partial U_3^0}{\partial Z}+\alpha^2h_X^2\frac{\partial U_3^1}{\partial Z}\Big) \\
&= \alpha^2\Big(\frac{\partial U_1^0}{\partial X}+h_X\frac{\partial U_1^0}{\partial Z}+h_X\frac{\partial U_3^0}{\partial X}+h_X^2\frac{\partial U_3^0}{\partial Z}\Big).
\end{aligned} \tag{180}$$

Since $U_3^0=0$, we deduce

$$\begin{aligned}
B &= \alpha^2\Big(\frac{\partial U_1^0}{\partial X}+h_X\frac{\partial U_1^0}{\partial Z}\Big) \\
&= \alpha^2\Big(\frac{-\sigma_0}{\mu}Zh_{XX}+h_X(\frac{-\sigma_0}{\mu}h_X)\Big) \\
&= -\alpha^2\frac{\sigma_0}{\mu}\Big(Zh_{XX}+h_X^2\Big).
\end{aligned} \tag{181}$$

We know that on $\Gamma_f(\tau)$, $Z=h$ thus B becomes

$$B = -\alpha^2\frac{\sigma_0}{\mu}\Big(hh_{XX}+h_X^2\Big), \tag{182}$$

and

$$K' = \frac{-f''}{(1+f'^2)^{\frac{3}{2}}} = \frac{-\alpha^2 h_{XX}}{(1+\alpha^2 h_X^2)^{\frac{3}{2}}}, \tag{183}$$

$$\begin{aligned} \delta E &= \frac{-1}{2}\frac{\sigma_0^2 f'^2\Big(f'^2\mu + (\lambda+2\mu)\Big)}{\mu(\lambda+2\mu)(1+f'^2)^2} - \frac{B\sigma_0\mu(\lambda+2\mu-\lambda f'^2) - 2\mu^2(\lambda+\mu)B^2}{\mu(\lambda+2\mu)(1+f'^2)^2} \\ &= \frac{-1}{2}\frac{\sigma_0^2\alpha^2 h_X^2\Big(\mu\alpha^2 h_X^2 + (\lambda+2\mu)\Big)}{\mu(\lambda+2\mu)(1+\alpha^2 h_X^2)^2} + \frac{\alpha^2\sigma_0^2(hh_{XX}+h_X^2)(\lambda+2\mu-\lambda\alpha^2 h_X^2)}{\mu(\lambda+2\mu)(1+\alpha^2 h_X^2)^2} \\ &\quad + \frac{2(\lambda+\mu)\alpha^4\sigma_0^2(hh_{XX}+h_X^2)^2}{\mu(\lambda+2\mu)(1+\alpha^2 h_X^2)^2}. \end{aligned} \tag{184}$$

While taking that the term of order α^2, it remains

$$\begin{aligned} \delta E &= \alpha^2\left(\frac{-1}{2}\frac{\sigma_0^2 h_X^2(\lambda+2\mu)}{\mu(\lambda+2\mu)(1+\alpha^2 h_X^2)^2} + \frac{\sigma_0^2(\lambda+2\mu)(hh_{XX}+h_X^2)}{\mu(\lambda+2\mu)(1+\alpha^2 h_X^2)^2}\right) \\ &= \alpha^2\left(\frac{-1}{2}\frac{\sigma_0^2}{\mu}h_X^2 + \frac{\sigma_0^2}{\mu}(hh_{XX}+h_X^2)\right) \\ &= \alpha^2\left[\frac{1}{2}\frac{\sigma_0^2}{\mu}h_X^2 + \frac{\sigma_0^2}{\mu}hh_{XX}\right] \\ &= \frac{\alpha^2\sigma_0^2}{\mu}(hh_{XX} + \frac{1}{2}h_X^2). \end{aligned} \tag{185}$$

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Captions to the figures

Figure 1 : Time evolution for the solution $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)). The initial pattern is plotted in dotted line. The represented times are (in timestep unit) 5, 10, 15, 20, 25, 30, 35, 45, 55, 65, 75, 85, 95, 105 and 115, respectively.

Figure 2 : Time evolution (continued) for the solution $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)). The represented times are now (in timestep unit) 135 to 139.

Figure 3 : Solutions $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)) at times 0.140 (top) and 0.141 (bottom).

Figure 4 : Solutions $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)) at times 0.143 (top) and 0.144 (bottom). Notice the huge difference in the amplitudes separated by a single time step, indicating that the solution is blowing up. The computation was interrupted at time 0.144.

Figure 5 : Zoom of the solutions $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)) at times 0.139 (top) and 0.144 (bottom). The collocation points are marked as symbols (+). Notice that even in the vicinity of the visual "spikes" of figures 2 or 4, the solution remains very smooth. The number of collocation points hence appears to be sufficient to correctly describe the solution.

Figure 6 : Initial positive pattern for the second round of computations, given by (149) and $M^r = 2.4$, $\alpha_0^r = 0.4$.

Figure 7 : Early time evolution of the solution to system (38) with an initial condition (dotted line) given by (149) and $M^r = 2.4$, $\alpha_0^r = 0.4$.

Figure 8 : Last computed pattern at time 0.083. Same run as in figure 7.

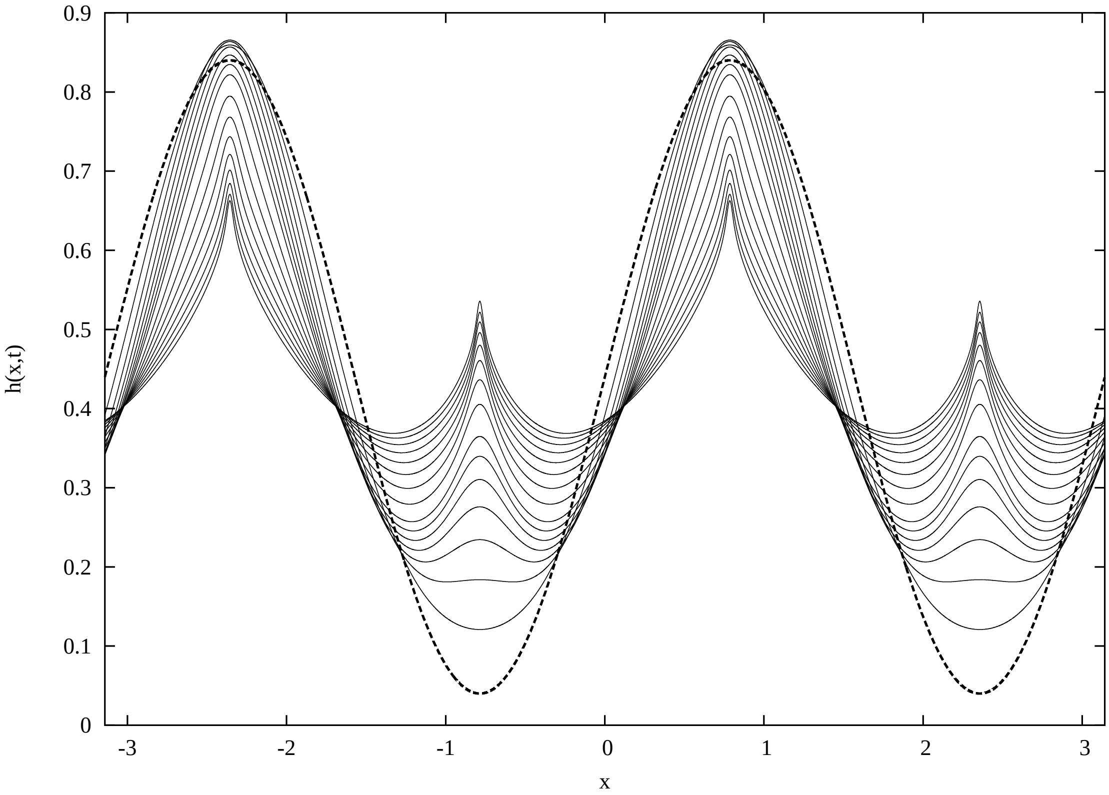

Figure 1: Time evolution for the solution $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)). The initial pattern is plotted in dotted line. The represented times are (in timestep unit) 5, 10, 15, 20, 25, 30, 35, 45, 55, 65, 75, 85, 95, 105 and 115, respectively.

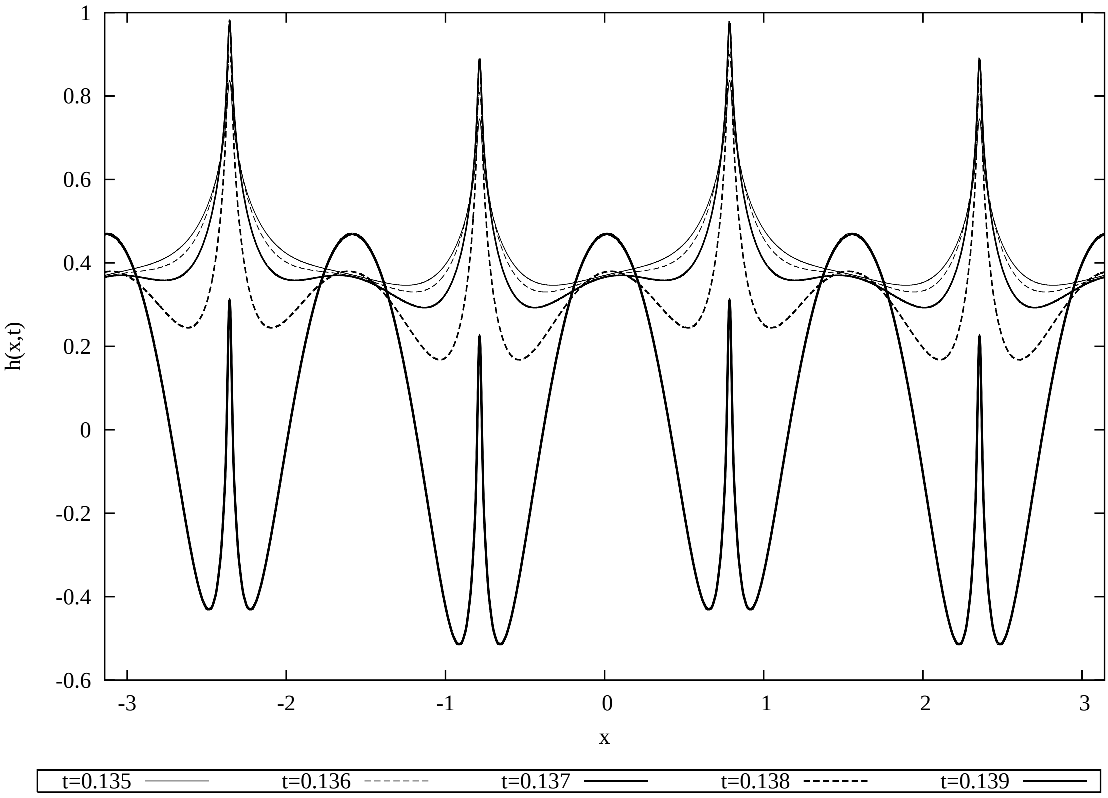

Figure 2: Time evolution (continued) for the solution $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)). The represented times are now (in timestep unit) 135 to 139.

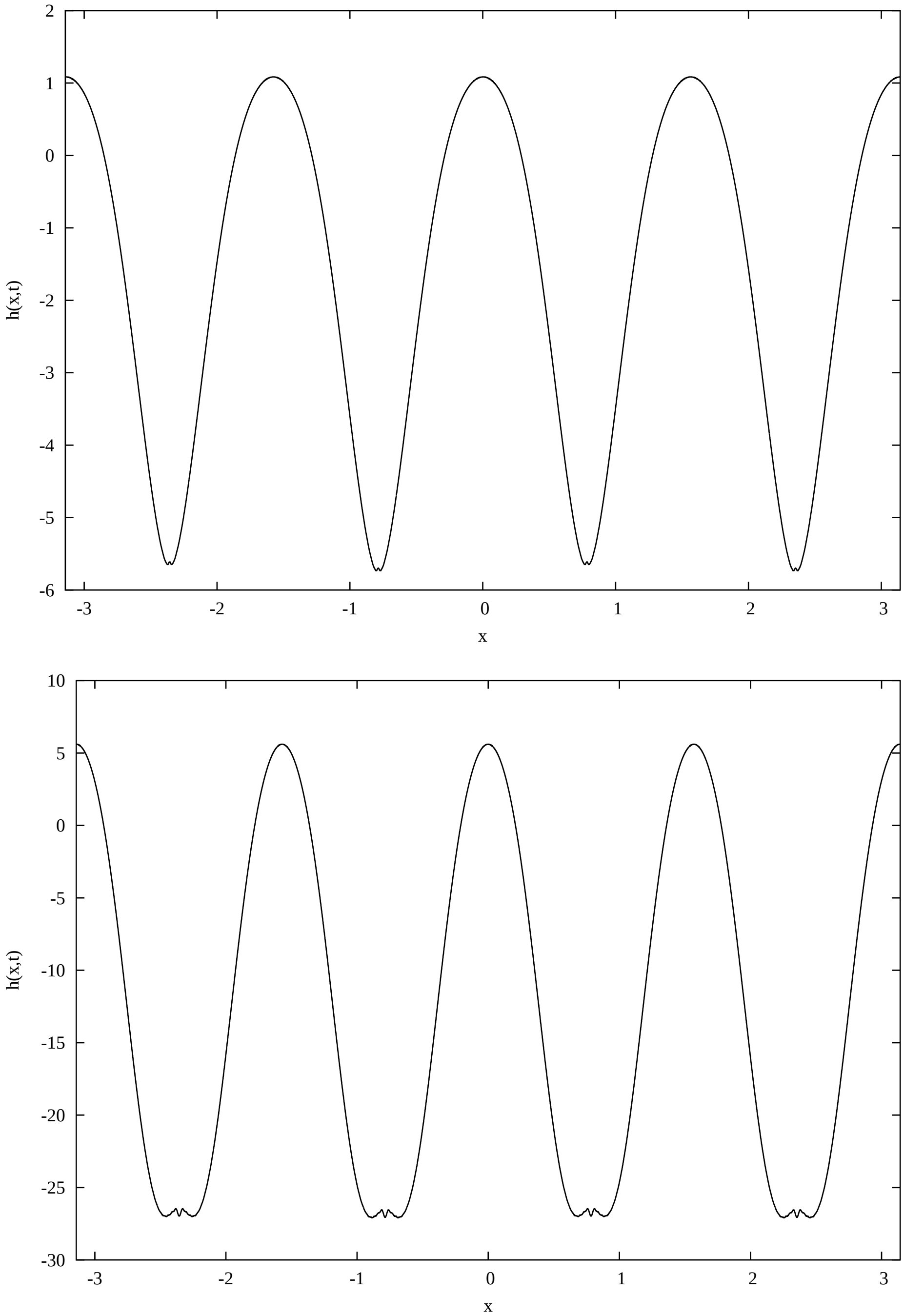

Figure 3: Solutions $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)) at times 0.140 (top) and 0.141 (bottom).

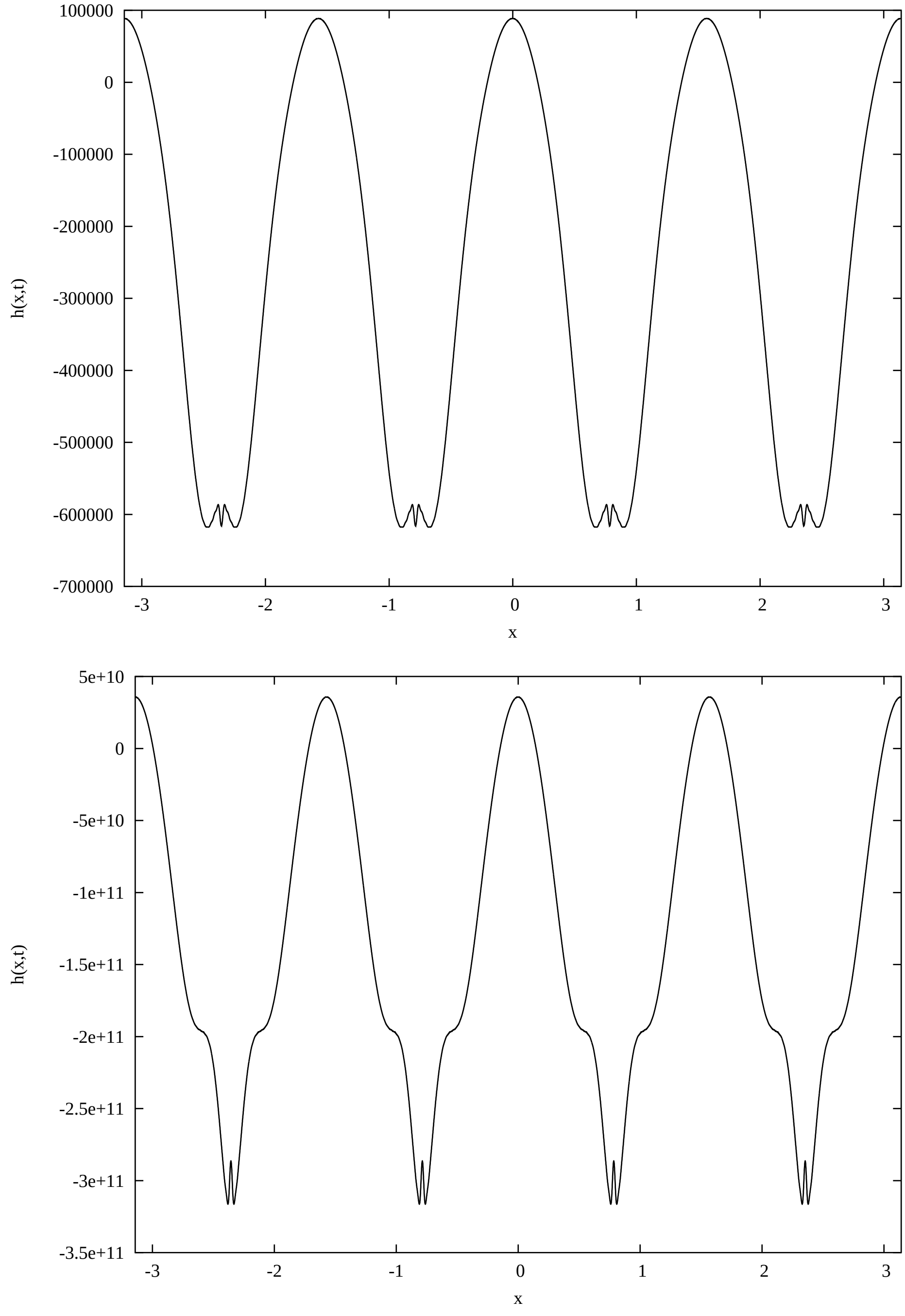

Figure 4: Solutions $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)) at times 0.143 (top) and 0.144 (bottom). Notice the huge difference in the amplitudes separated by a single time step, indicating that the solution is blowing up. The computation was interrupted at time 0.144.

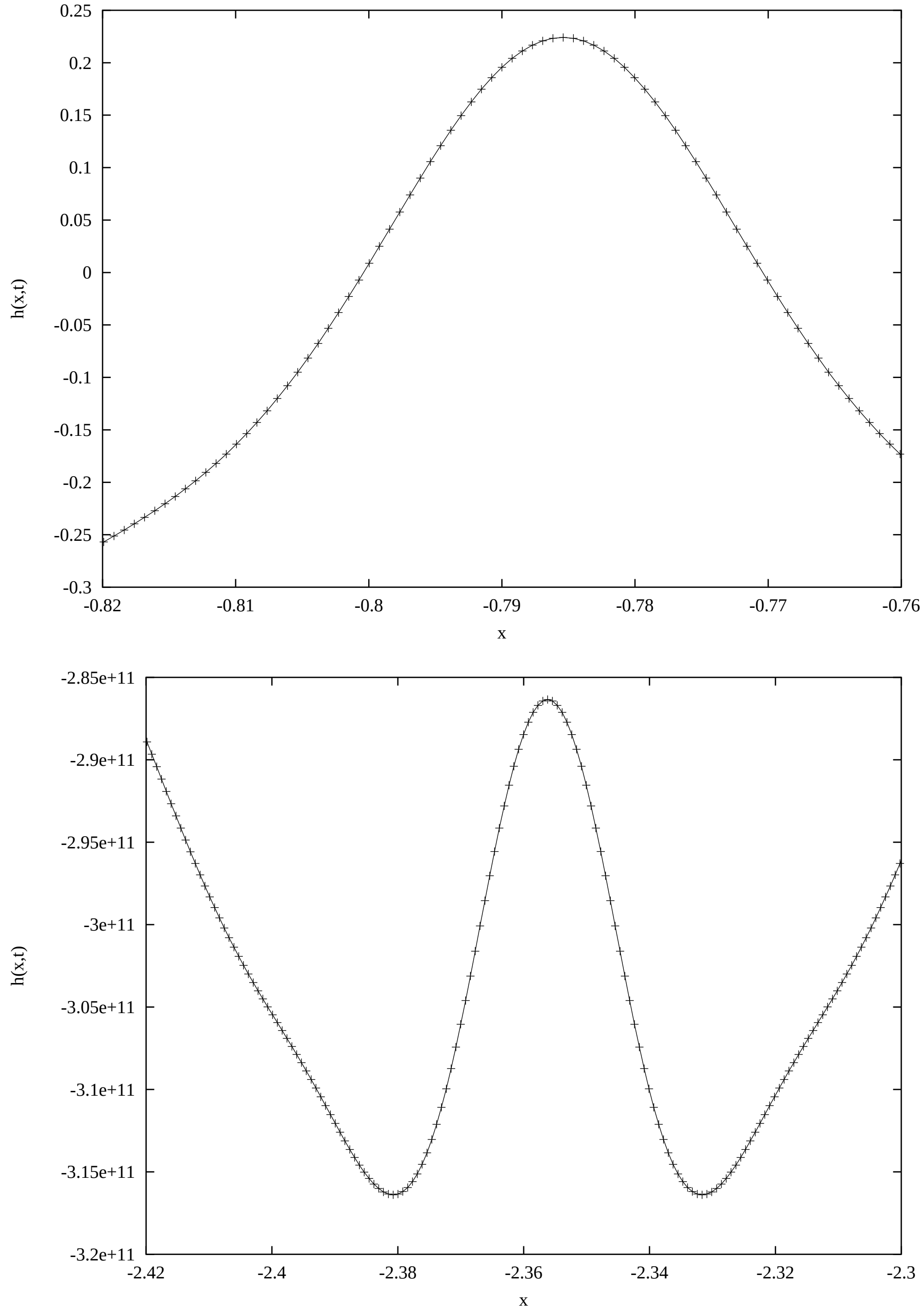

Figure 5: Zoom of the solutions $h(x,t)$ to system (38) with a sinusoidal positive initial condition (equation (148)) at times 0.139 (top) and 0.144 (bottom). The collocation points are marked as symbols (+). Notice that even in the vicinity of the visual “spikes” of figures 2 or 4, the solution remains very smooth. The number of collocation points hence appears to be sufficient to correctly describe the solution.

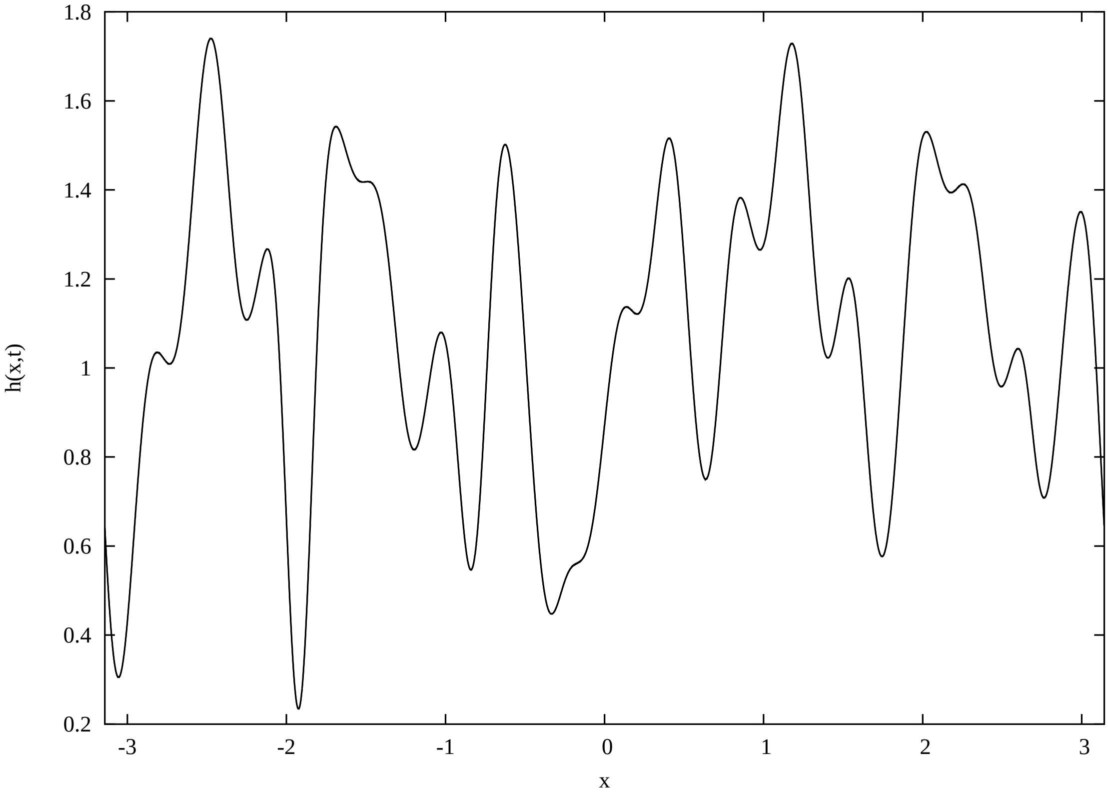

Figure 6: Initial positive pattern for the second round of computations, given by (149) and $M^r = 2.4$, $\alpha_0^r = 0.4$.

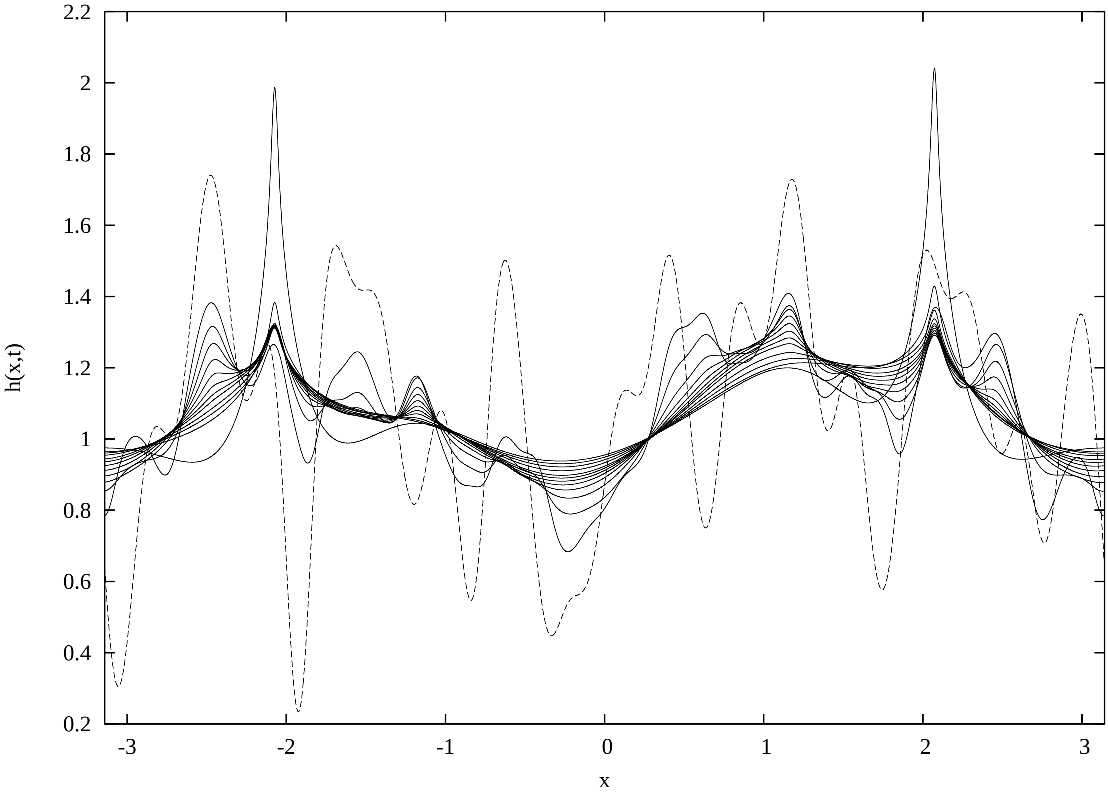

Figure 7: Early time evolution of the solution to system (38) with an initial condition (dotted line) given by (149) and $M^r = 2.4$, $\alpha_0^r = 0.4$.

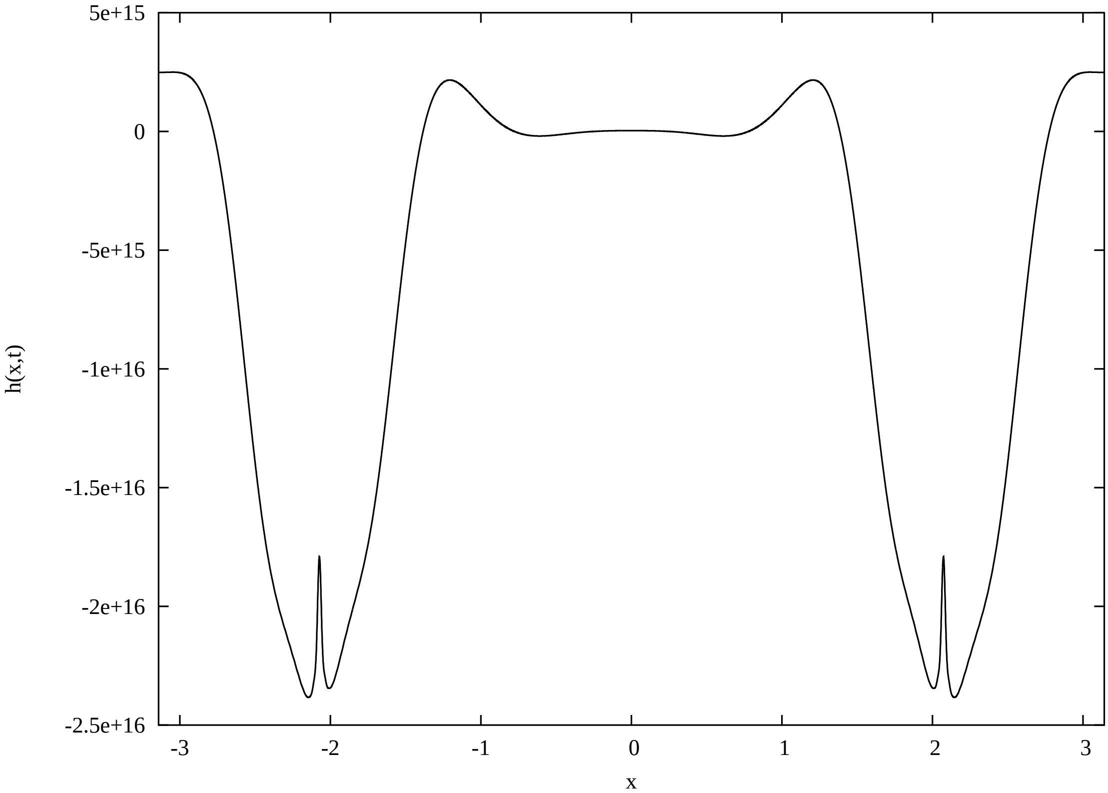

Figure 8: Last computed pattern at time 0.083. Same run as in figure 7.