Directional Lipschitzian Optimal Solution In Infinite–dimensional Optimization Problems

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Abstract : This paper presents a study of the Lipschitz dependence of the optimal solution of elementary convex programs in a Hilbert space when the equality constraints are subjected to small perturbations in some fixed direction and with the sub and super quadratic growth conditions. This study follows the recent results of Janin and Gauvin [9] related to the finite dimentional case. As an illustrative example, we study the directional derivative with respect to the boundary conditions of the infimum (value funtion) of the Mossolov problem in space dimension one.

Keywords : Nonsmooth analysis, optimization, optimality condition, value function, Green's function.

1. Introduction

Following the methods introduced in [9] by Janin and Gauvin in finite dimensional spaces, we consider the problem of the Lipschitz dependence of the optimal solution of an elementary convex problem with a nonsmooth convex objective function in an infinite dimensional Hilbert space when the equality constraints are subjected to small perturbations in some fixed direction. As in [9], these results are obtained assuming a sub and super quadratic growth conditions. As an illustrative example, we then study the directional derivative with respect to the boundary conditions of the infimum (value funtion) of the Mossolov problem in space dimension one [5, 11] in $H^1(]0,1[)$.
Let $g : E \longrightarrow I\!R \cup \{+\infty\}$ be a convex function, where E is a Hilbert space endowed with the scalar product (.,.). The polar (or conjugate) function of g is $g^*(y) = \sup\{(x,y) - g(x)/x \in E\}, \forall y \in E$.
We denote by $\partial g(x)$ the subdifferential of g at the point x. It is known that :

$$\partial g(x) = \{y \in E/g^*(y) + g(x) = (y,x)\}.$$

Let H and W be two Hilbert spaces. We denote by $(.,.)$ the scalar product on H and W, and by $|.|$ the associated norm. Let f be some convex function on H and A

be some linear operator from H onto W and consider the problem :

$$\text{Minimize } f(x) \text{ subject to } Ax = y. \tag{1}$$

Let v be a value (or marginal) function of (1) defined by :

$$v(y) = \inf\{f(x)/Ax = y\} \text{ for every } y \in W.$$

We assume the following :
H1: (super-quadratic growth condition)
$\exists \rho > 0$ such that $\forall x \in H$, $f(x) \geq f(x_0) + Df(x_0, x - x_0) + \frac{\rho}{2}|x - x_0|^2$.
H2: (sub-quadratic growth condition)
$\exists R > 0$ such that $\forall x \in H$, $f(x) \leq f(x_0) + Df(x_0, x - x_0) + \frac{R}{2}|x - x_0|^2$,
where $Df(x_0, h) = \lim_{t \searrow 0} \frac{1}{t}(f(x_0 + th) - f(x_0))$, and x_0 is an optimal solution of (1) for the perturbation $y = y_0$.
It is known that :

$$\partial v(y_0) = \{\eta \in W/\, A^*\eta \in \partial f(x_0)\}. \tag{2}$$

Since x_0 is optimal, a first-order necessary and sufficient optimality condition is :

$$\partial f(x_0) \cap A^*(W) \neq \emptyset.$$

First-order results are well-known in convex and nonconvex programs [8, 7, 12]. Second-order terms are evaluated in [2, 3]. Our aim in this article is to give some accurate bounds for the second-order term of the power series of the value function at y_0.
The following set will be important in this work

$$\sum(f, h) = \{\sigma \in \partial f(x_0), (\sigma, A^\sharp h) > Dv(y_0, h)\}, \tag{3}$$

where $A^\sharp = A^*(AA^*)^{-1}$ denotes the pseudo-inverse of A, A^* being the adjoint of A and h some fixed direction in W.
We denote by sgn the sign function on $\mathbb{R}$ defined at 0 by $sgn 0 = 0$.

2. Second-order results

Consider the function $g(x) = w(x) + \frac{k}{2}|x|^2$, where $w(.)$ is the support function of the convex set $B = \partial w(0)$. The polar function of g is given by

$$g^*(p) = \frac{1}{2k} \inf\{|p - q|^2 \,/\, q \in B\}.$$

We are interested now in the expression of $(g^*\circ A^*)^*(y)$ for $y\in W$.

Lemma 2.1. *For every $\sigma_0\in B\cap A^*(W)$, we have :*

$$(g^*\circ A^*)^*(y)=(\sigma_0,A^\sharp y)+\frac{k}{2}|A^\sharp y|^2+\sup_{\sigma\in B}\left\{(\sigma-\sigma_0,A^\sharp y)-\frac{1}{2k}|(1_H-A^\sharp A)(\sigma-\sigma_0)|^2\right\},$$

where 1_H denotes the identity mapping of H.

Lemma 2.2. *We assume that $B\cap A^*(W)\neq\emptyset$. Then we have the following result :*
i) If $\sigma\in B,(\sigma,A^\sharp y)\ >\ D(g^*\circ A^*)^*(0,y)$, then :

$$\lim_{t\searrow 0}\frac{2}{t^2}\Big((g^*\circ A^*)^*(ty)-(g^*\circ A^*)^*(0)-tD(g^*\circ A^*)^*(0,y)\Big)=$$

$$=k\{|A^\sharp y|^2+\sup_{\sigma\in B}\left\{\frac{[(\sigma,A^\sharp y)-D(g^*\circ A^*)^*(0,y)]^2}{|(1_H-A^\sharp A)\sigma|^2},\ (\sigma,A^\sharp y)\ >\ D(g^*\circ A^*)^*(0,y)\}\right\}.$$

ii) If $\sigma\in B,(\sigma,A^\sharp y)\le D(g^*\circ A^*)^*(0,y)$, then :

$$\lim_{t\searrow 0}\frac{2}{t^2}\Big((g^*\circ A^*)^*(ty)-(g^*\circ A^*)^*(0)-tD(g^*\circ A^*)^*(0,y)\Big)=k|A^\sharp y|^2.$$

3. Lipschitz type stability

We now derive a stability result for the optimal solution of the problem (1), using the result of §2. We set $x_0=x(y_0)$ and we denote by $x(y)$ the optimal solution associated to y.

Theorem 3.1. *We have the following :*
i) Either $\sum(f,y)\neq\emptyset$, in which case :

$$\rho\Big\{|A^\sharp y|^2+\sup_{\sigma\in\sum(f,y)}\{\frac{[(\sigma,A^\sharp y)-Dv(0,y)]^2}{|(1_H-A^\sharp A)\sigma|^2}\}\Big\}\le$$

$$\lim_{t\searrow 0}\frac{2}{t^2}\Big(v(ty)-v(0)-tDv(0,y)\Big)\le$$

$$R\Big\{|A^\sharp y|^2+\sup_{\sigma\in\sum(f,y)}\{\frac{[(\sigma,A^\sharp y)-Dv(0,y)]^2}{|(1_H-A^\sharp A)\sigma|^2}\}\Big\},$$

ii) or $\sum(f,y)=\emptyset$, in which case :

$$\rho|A^\sharp y|^2\le\lim_{t\searrow 0}\frac{2}{t^2}\Big(v(ty)-v(0)-tDv(0,y)\Big)\le R|A^\sharp y|^2.$$

In finite dimensional, Lipschitz continuity for the optimal solutions has been studied by Aubin [1] in the nonsmooth convex case with small perturbations. Analogous results can also be found in [4]. Hölder, Lipschitz and differential properties of the optimal solutions of a nonlinear mathematical programming problem with perturbations in some fixed direction are developed in [6]. In infinite dimensional space, some results on stability and sensitivity analysis with respect to the parameter are discussed in [10]. The following theorem gives the Lipschitz dependence for the optimal solution when the constraint y_0 is subjected to a small perturbation in some direction h, this stability being characterized by the emptiness or the nonemptiness of the set defined by (3).

Theorem 3.2. *We have the following alternative :*
i) If $\sum(f,h) \neq \emptyset$*, then :* $\limsup_{t\searrow 0} \frac{|x(y_0+th)-x(y_0)|}{t} \leq$

$$\left(\frac{R}{\rho}\right)^{\frac{1}{2}}\Big(1+\sup_{\sigma\in\sum(f,h)} \frac{|\sigma-\sigma_0|^2}{|(1_H-A^\sharp A)(\sigma-\sigma_0)|^2}\Big)^{\frac{1}{2}} \parallel A^\sharp \parallel |h|,$$

where σ_0 *is a subgradient of* f *at* x_0 *such that :* $(\sigma_0, A^\sharp h) = Dv(y_0,h)$.
ii) If $\sum(f,h) = \emptyset$*, then :*

$$\limsup_{t\searrow 0} \frac{|x(y_0+th)-x(y_0)|}{t} \leq \sqrt{\frac{R}{\rho}} \parallel A^\sharp \parallel |h|.$$

4. Example

In this paragraph, we consider the Mossolov's problem :
Minimize $J(v) = \frac{\alpha}{2}\int_0^1 |v'(x)|^2dx + \beta\int_0^1 |v'(x)|dx - \int_0^1 F(x)v(x)dx$,
subject to : $Av = (v(0), v(1)) = \theta \in I\!R^2$, with $v \in H^1(]0,1[)$,
where α and β are strictly positive constants, $F \in L^2(]0,1[)$ given.
In two space dimensions, the trajectories of the viscous-plastic medium's particules will be rectilinear during their motion in a pipe and their velocity $v(x,y)$ will be parallel to the pipe's axis. Mossolov and Miasnikov [11] studied the existence of optimal solutions such that the constraint is vanishing on the boundary. In [5], Ekeland and Temam extend these results by considering the primal and dual problems. Here, we study the dependence of the optimum of Mossolov's problem with respect to the boundary conditions.
The value function of the Mossolov problem is

$$\mathcal{V}(\theta) = \inf\{J(u)/Au = (u(0), u(1)) = \theta\}.$$

We assume that u_0 is an optimal solution associated with $\theta_0 \in I\!\!R^2$, and we set

$$L^\infty_\beta(u_0) = \Big\{ v \in L^\infty / |v(t)| \leq \beta \, a.e \; in \, \{u'_0 = 0\} \, and \, v(t) = \beta sgn u'_0(t) \, a.e \; in \, \{u'_0 \neq 0\} \Big\}.$$

Proposition 4.1. *The subdifferential of J at $u_0 \in H^1(]0,1[)$ is given by the formula*

$$\partial J(u_0) = \Big\{ \varphi \in H^1 / \varphi(.) = -\frac{ch \, .}{sh\, 1}(F_0(1) + \int_0^1 sh\,(1-y)(w(y) + \alpha u'_0(y) + F_0(y))dy) + \int_0^{.} ch\,(.-y)(w(y) + \alpha u'_0(y) + F_0(y))dy \, , \; w \in L^\infty_\beta(u_0) \, , \; F_0(.) = \int_0^{.} F(t)dt \Big\}.$$

Idea of the proof. The proof is based on the resolution of the system

$$\begin{cases} -T'' + T = f \;\; with \, f \in L^2, \\ T(0) = T(1) = 0. \end{cases}$$

It is well known that the solution of this system is given by :
$T(x) = \int_0^1 G(x,y) f(y) dy$, where the Green's function G is given by :
$G(x,y) = \dfrac{1}{sh\,1}(sh\,x\,sh\,(1-y) \; - sh\,1\,sh\,(x-y)^+)$,
and we have the following lemma :

Lemma 4.1.

$$\partial \Phi(u_0) = \Big\{ \varphi \in H^1 / \varphi(.) = -\frac{ch \, .}{sh\,1} \int_0^1 sh\,(1-y)v(y)dy + \int_0^{.} ch\,(.-y)v(y)dy, v \in L^\infty_\beta(u_0) \Big\},$$

where $\Phi(u) = \beta \int_0^1 |u'(x)| dx$, *for* $u \in H^1(]0,1[)$.

Remark. Inplementing Corollary 4.1, we obtain the directional derivative of the value function of the Mossolov's problem

$$D\mathcal{V}(\theta_0, \theta) = \begin{cases} (-\frac{1}{th\,1} m_1 + \frac{1}{sh\,1} m_4)\theta_1 + (-F_0(1) + \frac{1}{sh\,1} m_2 - \frac{1}{th\,1} m_3)\theta_2 & if \; \theta_1 \geq 0, \theta_2 \geq 0, \\ (-\frac{1}{th\,1} m_2 + \frac{1}{sh\,1} m_3)\theta_1 + (-F_0(1) + \frac{1}{sh\,1} m_1 - \frac{1}{th\,1} m_4)\theta_2 & if \; \theta_1 \leq 0, \theta_2 \leq 0, \\ (-\frac{1}{th\,1} m_2 + \frac{1}{sh\,1} m_3)\theta_1 + (-F_0(1) + \frac{1}{sh\,1} m_2 - \frac{1}{th\,1} m_3)\theta_2 & if \; \theta_1 \leq 0, \theta_2 \geq 0, \\ (-\frac{1}{th\,1} m_1 + \frac{1}{sh\,1} m_4)\theta_1 + (-F_0(1) + \frac{1}{sh\,1} m_1 - \frac{1}{th\,1} m_4)\theta_2 & if \; \theta_1 \geq 0, \theta_2 \leq 0, \end{cases}$$

where $m_1 = \inf\limits_{v \in L^\infty_\beta(u_0)} B_v(0^+)$, $m_2 = \sup\limits_{v \in L^\infty_\beta(u_0)} B_v(0^+)$, $m_3 = \inf\limits_{v \in L^\infty_\beta(u_0)} B_v(1^-)$,
$m_4 = \sup\limits_{v \in L^\infty_\beta(u_0)} B_v(1^-)$, where

$$B_v(x) = \frac{ch\,x}{sh\,1}\Big(\int_0^1 sh\,(1-y)(v(y) + \alpha u'_0(y) + F_0(y))dy \Big) - \int_0^x ch\,(x-y)(v(y) + \alpha u'_0(y) + F_0(y))dy, \; \forall x \in]0,1[, \; v \in L^\infty_\beta(u_0).$$

Using the Proposition 4.1, we obtain the following theorem and the stability of the optimal solution of Mossolov's problem in some direction is given by case ii) of Theorem 3.2.

Theorem 4.1. *The following relation holds :* $\sum(J, \theta) = \emptyset$.

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